# Connection Formulae for Asymptotics of Solutions of the Degenerate Third Painlevé Equation. I

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27 November 2003

**Abstract**

The degenerate third Painlevé equation, $u''=\frac{(u')^2}{u}-\frac{u'}{\tau}+\frac{1}{\tau}(-8\varepsilon u^2+2ab)+\frac{b^2}{u}$, where $\varepsilon, b\in\mathbb{R}$, and $a\in\mathbb{C}$, and the associated tau-function are studied via the Isomonodromy Deformation Method. Connection formulae for asymptotics of the general as $\tau\to\pm 0$ and $\pm\mathrm{i}0$ solution and general regular as $\tau\to\pm\infty$ and $\pm\mathrm{i}\infty$ solution are obtained.

**2000 Mathematics Subject Classification.** 33E17, 34M40, 34M50, 34M55, 34M60

**Abbreviated Title.** Degenerate Third Painlevé Equation

**Key Words.** Asymptotics, Painlevé transcendents, isomonodromy deformations, tau-function, Schlesinger transformations, WKB method, Stokes phenomena

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# 1 Introduction

In this paper we study the degenerate third Painlevé equation,

$$u''=\frac{(u')^2}{u}-\frac{u'}{\tau}+\frac{1}{\tau}(-8\varepsilon u^2+2ab)+\frac{b^2}{u}, \tag{1}$$

where $u=u(\tau)$, the primes denote differentiation with respect to $\tau$ (or $t$: see below), $a$ and $b$ $(\neq 0)$ are $\mathbb{C}$-valued parameters, and $\varepsilon=\pm 1$. Originally, Equation (1) appeared as a special case of, and in the same context as, the complete third Painlevé equation,

$$y''=\frac{(y')^2}{y}-\frac{y'}{t}+\frac{1}{t}\left(\alpha y^2+\beta\right)+\gamma y^3+\frac{\delta}{y}, \tag{2}$$

in the studies of Painlevé; however, its particular significance was not distinguished at that time.

There is another relation between Equations (1) and (2) (which is why Equation (1) is called degenerate), namely, the following *double-scaling limit*,

$$\begin{gathered} t=\epsilon\tau, \qquad y(t) \underset{\substack{\epsilon\to 0\\ \tau=\mathcal{O}(1)}}{=} \epsilon u(\tau)+o(\epsilon), \\ \alpha=-\frac{8\varepsilon}{\epsilon^2}, \qquad \beta=2ab, \qquad \gamma=o(\epsilon^{-4}), \qquad \delta=b^2. \end{gathered} \tag{3}$$

The $o(\epsilon)$ term in the formula for $y(t)$ above also depends on $\tau$: it is supposed that the derivatives with respect to $\tau$ of this term are also of the order $o(\epsilon)$. Only a few of the formal properties of Equation (1), namely, the Hamiltonian structure and Bäcklund transformations, can be derived from those for Equation (2) as a straightforward consequence of the double-scaling limit (3), whereas its analytic and asymptotic properties require separate considerations. Recently, Equation (1) has appeared in a number of physical [1–4] and geometrical applications [5] (in contexts independent of Equation (2)) where knowledge of asymptotic properties of its solutions is of special importance.

The Bäcklund transformation for Equation (1) was obtained by Gromak [6]: he also proved [7] that the only algebraic solutions of the complete third Painlevé equation are rational functions of $\tau^{1/3}$. Actually, these functions are solutions of Equation (1) for $a\mathrm{i}=n\in\mathbb{Z}$ and arbitrary, non-vanishing values of $\varepsilon$ and $b$. For fixed values of $a\mathrm{i}=n$, $b$ and $\varepsilon$, there is exactly one algebraic solution of Equation (1) which is a multi-valued function with three branches. This solution can be obtained by applying $|n|$ Bäcklund transformations to the simplest solution of Equation (1), namely, $u=b^{2/3}\tau^{1/3}/2\varepsilon$ (for $a=0$). All other solutions are non-classical in the sense of Darboux-Umemura [8, 9], that is, the absence of invariant algebraic curves of the corresponding Hamiltonian vector field: this was recently proved by Ohyama [10].

The most efficacious approach for studying the asymptotic behaviour of general solutions of the Painlevé equations and, especially, connection formulae for their asymptotics, is the method of isomonodromic deformations [11–15], or a closely related technique based on a steepest descent-type analysis of the associated Riemann-Hilbert problem [16]. Equation (1), in the case $a=0$, was studied by the isomonodromy deformation method in [17], where asymptotics as $\tau\to 0$ and $\infty$, as well as the corresponding connection formulae, were obtained: that work was based on the study of isomonodromic deformations of a $3\times 3$ matrix linear ODE with two irregular singular points. At the same time, as noted in [18], there is another matrix linear ODE in terms of $2{\times}2$ matrices whose isomonodromy deformations are described by Equation (1) (with arbitrary $a$). In this work, the latter ODE is used for the analysis of Equation (1).

**Proposition 1.1 ([18]).** *The necessary and sufficient condition for the compatibility of the linear system*

$$\partial_\lambda\Phi(\lambda,\tau)=\mathcal{U}(\lambda,\tau)\Phi(\lambda,\tau), \qquad \partial_\tau\Phi(\lambda,\tau)=\mathcal{V}(\lambda,\tau)\Phi(\lambda,\tau), \tag{4}$$

*with*

$$\mathcal{U}(\lambda,\tau)=\tau\left(-\mathrm{i}\sigma_3-\frac{a\mathrm{i}}{2\tau\lambda}\sigma_3-\frac{1}{\lambda}\begin{pmatrix}0 & C(\tau)\\ D(\tau) & 0\end{pmatrix}+\frac{\mathrm{i}}{2\lambda^2}\begin{pmatrix}\sqrt{-A(\tau)B(\tau)} & A(\tau)\\ B(\tau) & -\sqrt{-A(\tau)B(\tau)}\end{pmatrix}\right),$$

$$\mathcal{V}(\lambda,\tau)=\left(-\mathrm{i}\lambda\sigma_3+\frac{a\mathrm{i}}{2\tau}\sigma_3-\begin{pmatrix}0 & C(\tau)\\ D(\tau) & 0\end{pmatrix}-\frac{\mathrm{i}}{2\lambda}\begin{pmatrix}\sqrt{-A(\tau)B(\tau)} & A(\tau)\\ B(\tau) & -\sqrt{-A(\tau)B(\tau)}\end{pmatrix}\right),$$

$\sigma_3=\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, $\partial_z:=\frac{\partial}{\partial z}$, *and differentiable, scalar-valued functions* $A(\tau)$, $B(\tau)$, $C(\tau)$, *and* $D(\tau)$, *is that* $A(\tau)$, $B(\tau)$, $C(\tau)$, *and* $D(\tau)$ *satisfy the following system of isomonodromy deformations:*

$$\begin{gathered}A'(\tau)=4C(\tau)\sqrt{-A(\tau)B(\tau)}, \qquad B'(\tau)=-4D(\tau)\sqrt{-A(\tau)B(\tau)},\\ (\tau C(\tau))'=2a\mathrm{i}C(\tau)-2\tau A(\tau), \qquad (\tau D(\tau))'=-2a\mathrm{i}D(\tau)+2\tau B(\tau),\\ \left(\sqrt{-A(\tau)B(\tau)}\right)'=2(A(\tau)D(\tau)-B(\tau)C(\tau)).\end{gathered}\tag{5}$$

*Proof.* Follows from the Frobenius compatibility condition, $\partial_\tau\partial_\lambda\Phi(\lambda,\tau)=\partial_\lambda\partial_\tau\Phi(\lambda,\tau)$. □

**Remark 1.1.** System (5) implies that when $A(\tau)\equiv 0\Rightarrow B(\tau)C(\tau)=0$, and, when $B(\tau)\equiv 0\Rightarrow A(\tau)D(\tau)=0$: for these cases, System (5) can be solved exactly; hence, one excludes these cases from further consideration. Hereafter, all explicit $\tau$ dependencies are suppressed, except where confusion may arise. ■

**Proposition 1.2.** *Let* $u=u(\tau)$ *and* $\varphi=\varphi(\tau)$ *solve the following system,*

$$\begin{gathered}u''=\frac{(u')^2}{u}-\frac{u'}{\tau}+\frac{1}{\tau}(-8\varepsilon u^2+2ab)+\frac{b^2}{u},\\ \varphi'=\frac{2a}{\tau}+\frac{b}{u},\end{gathered}\tag{6}$$

*where* $\varepsilon=\pm 1$, *and* $a,b\in\mathbb{C}$ *are independent of* $\tau$. *Then*

$$A(\tau):=\frac{u(\tau)}{\tau}\mathrm{e}^{\mathrm{i}\varphi(\tau)}, \quad B(\tau):=-\frac{u(\tau)}{\tau}\mathrm{e}^{-\mathrm{i}\varphi(\tau)}, \quad C(\tau):=\frac{\varepsilon\tau}{4u(\tau)}A'(\tau), \quad D(\tau):=-\frac{\varepsilon\tau}{4u(\tau)}B'(\tau),$$

*solve System* (5). *Conversely, let* $A(\tau)\not\equiv 0$, $B(\tau)\not\equiv 0$, $C(\tau)$, *and* $D(\tau)$ *solve System* (5), *and define*

$$u(\tau):=\varepsilon\tau\sqrt{-A(\tau)B(\tau)}, \qquad \varphi(\tau):=-\tfrac{\mathrm{i}}{2}\ln(-A(\tau)/B(\tau)), \qquad \text{and} \qquad b:=u(\tau)(\varphi'(\tau)-2a/\tau).$$

*Then* $b$ *is independent of* $\tau$, *and* $u(\tau)$ *solves Equation* (1).

*Proof.* Straightforward verification. □

**Remark 1.2.** For $b=0$, System (6) yields $u(\tau)=\frac{\varepsilon\widetilde{c}_1}{16\tau}-\frac{\varepsilon\widetilde{c}_1\left(1+\widetilde{c}_2\tau^{\sqrt{\widetilde{c}_1}}\right)^2}{16\tau\left(1-\widetilde{c}_2\tau^{\sqrt{\widetilde{c}_1}}\right)^2}$, where $\widetilde{c}_i\in\mathbb{C}$, $i=1,2$, and $\varphi(\tau)=2a\ln\tau+\varphi_o$, with $\varphi_o\in\mathbb{C}$; therefore, hereafter, only the case $b\neq 0$ will be considered. In the latter case, one can rescale the parameters $\varepsilon,b$ from Equation (1) via the following sequence of scaling transformations: (1) $\tau\to\tau/\sqrt{\varepsilon b}$ and $u\to u\sqrt{\varepsilon b}$; followed by (2) $\tau\to\varepsilon\tau$ and $u\to u$. As these algebraic rescalings change $\arg\tau$, which is important for asymptotics, it is convenient to keep the parameters $\varepsilon,b$ explicitly. ■

The Hamiltonian structure of Equation (1) can be derived from the corresponding structure of Equation (2) [12, 19] by applying the double-scaling limit (3).

**Proposition 1.3.** *Let*

$$\mathcal{H}_{\varepsilon_1}(p,q;\tau):=\frac{p^2q^2}{\tau}-\frac{2\varepsilon_1 pq(a\mathrm{i}+1/2)}{\tau}+4\varepsilon q+\mathrm{i}bp+\frac{(a\mathrm{i}+1/2)^2}{2\tau}, \tag{7}$$

*where* $\varepsilon^2=\varepsilon_1^2=1$. *Then Hamilton's equations,*

$$\frac{\mathrm{d}p}{\mathrm{d}\tau}=-\frac{\partial\mathcal{H}_{\varepsilon_1}(p,q;\tau)}{\partial q} \qquad \text{and} \qquad \frac{\mathrm{d}q}{\mathrm{d}\tau}=\frac{\partial\mathcal{H}_{\varepsilon_1}(p,q;\tau)}{\partial p}, \tag{8}$$

*are equivalent to either one of the following degenerate third Painlevé equations,*

$$p''=\frac{(p')^2}{p}-\frac{p'}{\tau}+\frac{1}{\tau}\Big(-2\mathrm{i}bp^2+8\varepsilon\Big(a\mathrm{i}\varepsilon_1+\tfrac{(\varepsilon_1-1)}{2}\Big)\Big)-\frac{16}{p}, \tag{9}$$

$$q''=\frac{(q')^2}{q}-\frac{q'}{\tau}+\frac{1}{\tau}\big(-8\varepsilon q^2-b(2a\varepsilon_1-\mathrm{i}(1+\varepsilon_1))\big)+\frac{b^2}{q}. \tag{10}$$

*Proof.* The Hamiltonian System (8) can be rewritten as

$$p=\frac{\tau(q'-\mathrm{i}b)}{2q^2}+\frac{(a\mathrm{i}+1/2)\varepsilon_1}{q}, \qquad q=-\frac{\tau(p'+4\varepsilon)}{2p^2}+\frac{(a\mathrm{i}+1/2)\varepsilon_1}{p}. \tag{11}$$

The proposition follows from Equation (11) by straightforward calculations. □

**Remark 1.3.** Either expression in Equation (11) is equivalent to the Bäcklund transformation [7] for Equation (1). The connection of $\mathcal{H}_{\varepsilon_1}(p,q;\tau)$ with the isomonodromy deformations is given in Remark 2.2 of Section 2. ■

It is worth noting two other differential equations related to the degenerate third Painlevé equation:

$$\tau^2\big(f''+4\mathrm{i}\varepsilon b\big)^2=(4f-\varepsilon_1(2\mathrm{i}a+1))^2\big((f')^2+8\mathrm{i}\varepsilon bf\big),$$

$$\big(\tau\sigma''-\sigma'\big)^2=2\big(2\sigma-\tau\sigma'\big)\big(\sigma'\big)^2-32\mathrm{i}\varepsilon b\tau\big(\big(\tfrac{1-\varepsilon_1}{2}-a\mathrm{i}\varepsilon_1\big)\sigma'+2\mathrm{i}\varepsilon b\tau\big),$$

where

$$f(\tau):=\frac{p(\tau)q(\tau)}{2},$$

and

$$\begin{aligned}\sigma(\tau) &:= \tau\mathcal{H}_{\varepsilon_1}(p(\tau),q(\tau);\tau)+2f(\tau)+\tfrac{1}{2}\big(a\mathrm{i}+\tfrac{1}{2}\big)^2-\varepsilon_1\big(a\mathrm{i}+\tfrac{1}{2}\big)+\tfrac{1}{4}\\ &= \big(p(\tau)q(\tau)-\varepsilon_1\big(a\mathrm{i}+\tfrac{1}{2}-\tfrac{\varepsilon_1}{2}\big)\big)^2+\tau(4\varepsilon q(\tau)+\mathrm{i}bp(\tau)).\end{aligned}$$

The functions $u(\tau)$ and $f(\tau)$ solve some well-known integrable differential-difference and difference equations which are given in Section 6. In this work, asymptotics of Equation (1) as $\tau\to\pm 0$, $\pm\mathrm{i}0$, $\pm\infty$, and $\pm\mathrm{i}\infty$ are parametrised in terms of the monodromy data of System (4). This parametrisation is equivalent to finding the corresponding connection formulae: indeed, given asymptotics of some solution as $\tau\to\pm 0$ (resp., $\pm\mathrm{i}0$) or $\pm\infty$ (resp., $\pm\mathrm{i}\infty$), it is straightforward to determine the corresponding monodromy data, and therefore to obtain asymptotics of the same solution as $\tau\to\pm\infty$ (resp., $\pm\mathrm{i}\infty$) or $\pm 0$ (resp., $\pm\mathrm{i}0$).

There are different technical means by which isomonodromy deformations can be used to obtain the desired parametrisation of asymptotics. For the first time, isomonodromy deformations for the solution of the connection problem for the six Painlevé equations were used by Jimbo [14]; however, that work only dealt with asymptotics of solutions of the Painlevé equations in the neighbourhood of the regular singular points. At about the same time, Its *et al.* [20] used isomonodromy deformations for the parametrisation of asymptotics of a special solution to the third Painlevé equation in the neighbourhood of its essential singular point by means of the corresponding monodromy data; however, this was done in the context of the

solution of an asymptotic problem for special (rapidly decaying) solutions of the sine-Gordon equation rather than the connection problem for the third Painlevé equation. Novokshenov [21] used similar methods to solve the connection problem for a particular solution of the third Painlevé equation. These asymptotic methods for the Painlevé equations (see, also, [22]), based on the asymptotic analysis of associated linear ODEs, were developed further in a number of works; for a recent example, see [23]. Here, we also follow these ideas.

Primary emphasis is given to the global asymptotic properties of $\Phi(\lambda,\tau)$, that is, the possibility of matching different local asymptotic expansions of $\Phi(\lambda,\tau)$ at singular and turning points, which culminates in the asymptotic expansion of solutions of the Painlevé equations. The justification scheme for this asymptotic method was suggested in [24]. Further details concerning its application to particular Painlevé equations can be found in [25, 26].

This paper is organized as follows. In Section 2, the principal object of study, that is, the manifold of monodromy data, is introduced. In Section 3, the results of this work, namely, asymptotics of general solutions of Equation (1) parametrised in terms of the monodromy data (points of the monodromy manifold), are given. In Section 4 (resp., Section 5), the $\tau\to+\infty$ (resp., $\tau\to+0$) asymptotic analysis for the general solutions of Equation (1) is presented. In Section 6, the Bäcklund transformations for Equation (1) are derived as a consequence of the Schlesinger transformations (cf. [18]). This allows one to remove the restriction on the parameter (of formal monodromy) $a$ which is imposed in Section 5, and to extend the corresponding asymptotic (and connection) results as $\tau\to+0$ to any complex value of the parameter $a$. In the same section we also find the actions of the Lie-point symmetries for System (5) on the manifold of the monodromy data: the latter is used to extend the connection results found in the previous sections for asymptotics on the positive semi-axis to asymptotics on the negative semi-axis, and, in the Appendix, for asymptotics on the imaginary axis.

Many modern applications actually require knowledge of the function $\mathcal{H}(\tau)$ rather than the original Painlevé function $u(\tau)$. Therefore, we present the corresponding asymptotic results for the function $\mathcal{H}(\tau)$. Actually, our asymptotics for the function $\mathcal{H}(\tau)$ as $\tau\to\pm\infty$ is "incomplete" as the leading term that directly follows from our calculations is parametrised by only one monodromy parameter, while there should be two. To obtain a complete parametrisation, one has to find one more term in the asymptotic expansion of $\mathcal{H}(\tau)$: the latter requires the calculation of two further terms in the asymptotic expansion for the function $u(\tau)$. This calculation is not related to the isomonodromy deformation technique considered here, but can be obtained by a direct substitution of the corresponding ansatz into Equation (1). Our asymptotic results as $\tau\to\pm\infty$ are also incomplete in another sense: we haven't found asymptotics for all domains of the monodromy manifold; in particular, asymptotics as $\tau\to\pm\infty$ of the singular real solutions of Equation (1). We are planning on closing both "gaps" mentioned above in a subsequent work.

## 2 The Manifold of Monodromy Data

The leading terms, as $\lambda\to0$, of the matrices $\mathcal{U}(\lambda,\tau)$ and $\mathcal{V}(\lambda,\tau)$ in System (4) are degenerate. Therefore, we begin with a Fabry-type transformation (see [27]) to obtain, instead of System (4), one with non-degenerate leading terms at irregular singular points.

**Proposition 2.1.** *Define variables $\mu$ and $\Psi=\Psi(\mu,\tau)$ :*

$$\lambda=\mu^2, \qquad \Phi(\lambda,\tau):=\sqrt{\mu}\left(\begin{pmatrix}1&0\\0&0\end{pmatrix}+\frac{1}{\mu}\begin{pmatrix}0&-\frac{A}{\sqrt{-AB}}\\0&1\end{pmatrix}\right)\Psi(\mu,\tau),$$

*where $\Phi(\lambda,\tau)$ is a fundamental solution of System (4). Then*

$$\partial_\mu\Psi=\widetilde{\mathcal{U}}(\mu,\tau)\Psi, \qquad \partial_\tau\Psi=\widetilde{\mathcal{V}}(\mu,\tau)\Psi, \tag{12}$$

*where*

$$\widetilde{\mathcal{U}}(\mu,\tau)=-2\mathrm{i}\tau\mu\sigma_3+2\tau\begin{pmatrix}0&\frac{2\mathrm{i}A}{\sqrt{-AB}}\\-D&0\end{pmatrix}-\frac{1}{\mu}\left(a\mathrm{i}+\frac{2\tau AD}{\sqrt{-AB}}+\frac{1}{2}\right)\sigma_3+\frac{1}{\mu^2}\begin{pmatrix}0&\widetilde{\alpha}\\\mathrm{i}\tau B&0\end{pmatrix},$$

$$\widetilde{\mathcal{V}}(\mu,\tau)=-\mathrm{i}\mu^2\sigma_3+\mu\begin{pmatrix}0 & \frac{2\mathrm{i}A}{\sqrt{-AB}}\\ -D & 0\end{pmatrix}+\left(\frac{a\mathrm{i}}{2\tau}-\frac{AD}{\sqrt{-AB}}\right)\sigma_3-\frac{1}{2\tau\mu}\begin{pmatrix}0 & \widetilde{\alpha}\\ \mathrm{i}\tau B & 0\end{pmatrix},$$

*with*

$$\widetilde{\alpha}:=-\frac{2}{B}\Big(a\mathrm{i}\sqrt{-AB}+\tau(AD+BC)\Big). \tag{13}$$

*Proof.* Follows from the change of independent and dependent variables given in the Proposition and System (4). □

**Lemma 2.1.**

$$\frac{1}{\tau}\det\begin{pmatrix}0 & \widetilde{\alpha}\\ \mathrm{i}\tau B & 0\end{pmatrix}=-\mathrm{i}\widetilde{\alpha}B=\varepsilon b,\quad \varepsilon=\pm1, \tag{14}$$

*with* $\widetilde{\alpha}$ *defined by Equation* (13).

*Proof.* Substituting the parametrisation for $A$, $B$, $C$, and $D$ in terms of $u(\tau)$ and $\varphi(\tau)$ (cf. Proposition 1.2), one deduces that $\widetilde{\alpha}B=\mathrm{i}\varepsilon u(\varphi'-2a/\tau)$: the result stated in the Lemma now follows from Equation (6). □

System (12) has two irregular singular points, $\mu=\infty$ and $\mu=0$. For $\delta>0$ and $k\in\mathbb{Z}$, define the (sectorial) neighbourhoods $\Omega_k^{\infty}$ and $\Omega_k^0$, respectively, of these points:

$$\Omega_k^{\infty}:=\left\{\mu;\ |\mu|>\delta^{-1},\ -\tfrac{\pi}{2}+\tfrac{\pi k}{2}<\arg\mu+\tfrac{1}{2}\arg\tau<\tfrac{\pi}{2}+\tfrac{\pi k}{2}\right\},$$
$$\Omega_k^{0}:=\left\{\mu;\ |\mu|<\delta,\ -\pi+\pi k<\arg\mu-\tfrac{1}{2}\arg\tau-\tfrac{1}{2}\arg(\varepsilon b)<\pi+\pi k\right\}.$$

The following Proposition is a direct consequence of general asymptotic results for linear ODEs [28, 29] and Lemma 2.1.

**Proposition 2.2.** *For* $k\in\mathbb{Z}$*, there exist solutions* $Y_k^{\infty}(\mu)$ *and* $X_k^0(\mu)$ *of System* (12) *which are uniquely defined by the following asymptotic expansions:*

$$Y_k^{\infty}(\mu)\underset{\substack{\mu\to\infty\\ \mu\in\Omega_k^{\infty}}}{:=}\left(\mathrm{I}+\frac{1}{\mu}\Psi^{(1)}+\frac{1}{\mu^2}\Psi^{(2)}+\cdots\right)\exp\!\big(-\mathrm{i}(\tau\mu^2+(a-\mathrm{i}/2)\ln\mu)\sigma_3\big), \tag{15}$$

$$X_k^0(\mu)\underset{\substack{\mu\to0\\ \mu\in\Omega_k^{0}}}{:=}\Psi_0(\mathrm{I}+\mathcal{Z}_1\mu+\cdots)\exp\left(-\frac{\mathrm{i}\sqrt{\tau\varepsilon b}}{\mu}\sigma_3\right), \tag{16}$$

*where* $\ln\mu:=\ln|\mu|+\mathrm{i}\arg\mu$,

$$\mathrm{I}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad \Psi^{(1)}=\begin{pmatrix}0 & \frac{A}{\sqrt{-AB}}\\ \frac{D}{2\mathrm{i}} & 0\end{pmatrix},\qquad \Psi^{(2)}=\begin{pmatrix}\psi_{11}^{(2)} & 0\\ 0 & \psi_{22}^{(2)}\end{pmatrix}, \tag{17}$$
$$\psi_{11}^{(2)}=-\tfrac{\mathrm{i}}{2}\left(\tau\sqrt{-AB}+\tau DC+\tfrac{AD}{\sqrt{-AB}}\right),\qquad \psi_{22}^{(2)}=\tfrac{\mathrm{i}\tau}{2}\left(\sqrt{-AB}+CD\right),$$

$$\Psi_0=\tfrac{\mathrm{i}}{\sqrt{2}}\left(\tfrac{(\varepsilon b)^{1/4}}{\tau^{1/4}\sqrt{B}}\right)^{\sigma_3}(\sigma_1+\sigma_3),\qquad \mathcal{Z}_1=\begin{pmatrix}z_1^{(11)} & z_1^{(12)}\\ -z_1^{(12)} & -z_1^{(11)}\end{pmatrix}, \tag{18}$$
$$z_1^{(11)}=\frac{\left(a\mathrm{i}+\frac{1}{2}+\frac{2\tau AD}{\sqrt{-AB}}\right)^2}{2\mathrm{i}\sqrt{\tau\varepsilon b}}-\frac{2\mathrm{i}\tau^{3/2}\sqrt{-AB}}{\sqrt{\varepsilon b}}-\frac{D\sqrt{\tau\varepsilon b}}{B},\qquad z_1^{(12)}=\frac{\left(a\mathrm{i}+\frac{1}{2}+\frac{2\tau AD}{\sqrt{-AB}}\right)}{2\mathrm{i}\sqrt{\tau\varepsilon b}}.$$

In Proposition 2.2 and below, we use the Pauli matrices

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad \sigma_2=\begin{pmatrix}0&-\mathrm{i}\\ \mathrm{i}&0\end{pmatrix},\qquad\text{and}\qquad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

The *canonical solutions*, $Y_k^{\infty}(\mu)$ and $X_k^0(\mu)$, enable one to define the *Stokes matrices* $S_k^{\infty}$ and $S_k^0$:

$$Y_{k+1}^{\infty}(\mu)=Y_k^{\infty}(\mu)S_k^{\infty}, \tag{19}$$

$$X^0_{k+1}(\mu)=X^0_k(\mu)S^0_k. \tag{20}$$

The Stokes matrices are independent of the parameters $\mu$ and $\tau$, and have the following structures:

$$S^\infty_{2k}=\begin{pmatrix}1 & 0\\ s^\infty_{2k} & 1\end{pmatrix}, \quad S^\infty_{2k+1}=\begin{pmatrix}1 & s^\infty_{2k+1}\\ 0 & 1\end{pmatrix}, \quad S^0_{2k}=\begin{pmatrix}1 & s^0_{2k}\\ 0 & 1\end{pmatrix}, \quad S^0_{2k+1}=\begin{pmatrix}1 & 0\\ s^0_{2k+1} & 1\end{pmatrix}. \tag{21}$$

The parameters $s^\infty_n$ and $s^0_n$, $n\in\mathbb{Z}$, are called the *Stokes multipliers*. Using Equations (15) and (16), one shows that

$$Y^\infty_{k+4}(\mu e^{2\pi i})=Y^\infty_k(\mu)e^{2\pi(a-i/2)\sigma_3}, \qquad X^0_{k+2}(\mu e^{2\pi i})=X^0_k(\mu); \tag{22}$$

hence, from Equations (19), (20), and (22), one arrives at

$$S^\infty_{k+4}=e^{-2\pi(a-i/2)\sigma_3}S^\infty_k e^{2\pi(a-i/2)\sigma_3}, \qquad S^0_{k+2}=S^0_k. \tag{23}$$

Equations (23) show that the number of independent Stokes multipliers does not exceed six, e.g., $s^0_0$, $s^0_1$, $s^\infty_0$, $s^\infty_1$, $s^\infty_2$, and $s^\infty_3$. Furthermore, due to the special structure of System (12), namely, the coefficient matrices of odd (resp., even) powers of $\mu$ in $\widetilde{\mathcal{U}}(\mu,\tau)$ are diagonal (resp., off-diagonal) and *vice-versa* for $\widetilde{\mathcal{V}}(\mu,\tau)$, one can deduce further relations between the Stokes multipliers. More precisely, as a result of the above-mentioned special structure, one obtains the following symmetry reductions for the canonical solutions:

$$Y^\infty_k(\mu)=\sigma_3 Y^\infty_{k+2}(\mu e^{\pi i})\sigma_3 e^{-\pi(a-i/2)\sigma_3}, \qquad X^0_k(\mu)=\sigma_3 X^0_{k+1}(\mu e^{\pi i})\sigma_1. \tag{24}$$

Equations (24) imply the following relations for the Stokes matrices (multipliers):

$$S^\infty_{k+2}=\sigma_3 e^{-\pi(a-i/2)\sigma_3}S^\infty_k e^{\pi(a-i/2)\sigma_3}\sigma_3, \qquad S^0_k=\sigma_1 S^0_{k+1}\sigma_1. \tag{25}$$

Equations (25) reduce the number of independent Stokes multipliers by a factor of 2; in particular, all Stokes multipliers can be expressed in terms of $s^0_0$, $s^\infty_0$, $s^\infty_1$, and the parameter of formal monodromy, $a$.

There is one more relation between the Stokes multipliers which follows from the so-called cyclic relation. Define the monodromy matrices at infinity, $M^\infty$, and at zero, $M^0$, by the following relations:

$$Y^\infty_0(\mu e^{-2\pi i}):=Y^\infty_0(\mu)M^\infty, \tag{26}$$

$$X^0_0(\mu e^{-2\pi i}):=X^0_0(\mu)M^0. \tag{27}$$

Since $Y^\infty_0(\mu)$ and $X^0_0(\mu)$ are solutions of System (12), they differ by a right-hand factor $G$,

$$Y^\infty_0(\mu):=X^0_0(\mu)G, \tag{28}$$

where $G$ is called the *connection matrix*. As matrices relating fundamental solutions of System (12), the monodromy, connection, and Stokes matrices are independent of $\mu$ and $\tau$. Furthermore, since $\operatorname{tr}(\widetilde{\mathcal{U}}(\mu,\tau))=\operatorname{tr}(\widetilde{\mathcal{V}}(\mu,\tau))=0$ (cf. Proposition 2.1, System (12)), it follows that

$$\det(M^0)=\det(M^\infty)=\det(G)=1. \tag{29}$$

From the definition of the monodromy and connection matrices, one deduces the following *cyclic relation*:

$$GM^\infty=M^0G. \tag{30}$$

The monodromy matrices can be expressed in terms of the Stokes matrices:

$$M^\infty=S^\infty_0 S^\infty_1 S^\infty_2 S^\infty_3 e^{-2\pi(a-i/2)\sigma_3}, \qquad M^0=S^0_0 S^0_1. \tag{31}$$

Due to the symmetry reductions (24), one can derive, using Equations (19), (20), and (28), the *semi-cyclic* relation:

$$G^{-1}S_0^0\sigma_1 G = S_0^\infty S_1^\infty \sigma_3 \mathrm{e}^{-\pi(a-\mathrm{i}/2)\sigma_3}. \tag{32}$$

Using the first of Equations (25), one shows that $M^\infty = (S_0^\infty S_1^\infty \sigma_3 \mathrm{e}^{-\pi(a-\mathrm{i}/2)\sigma_3})^2$; hence, the semi-cyclic relation (Equation (32)) implies the cyclic one (Equation (30)). The Stokes multipliers, $s_0^0$, $s_0^\infty$, and $s_1^\infty$, the elements of the connection matrix, $(G)_{ij} =: g_{ij}$, $i,j=1,2$, and the parameter of formal monodromy, $a$, are called the *monodromy data*. Consider $\mathbb{C}^8$ with co-ordinates $(a, s_0^0, s_0^\infty, s_1^\infty, g_{11}, g_{12}, g_{21}, g_{22})$. The algebraic variety defined by $\det(G)=1$ and Equation (32) is called the *manifold of monodromy data*, $\mathcal{M}$. Since only three of the four equations in (32) are independent, it is clear that $\dim_{\mathbb{C}}(\mathcal{M})=4$. More precisely, the equations defining $\mathcal{M}$ are as follows:

$$\begin{gathered} s_0^\infty s_1^\infty = -1-\mathrm{e}^{-2\pi a}-\mathrm{i}s_0^0\mathrm{e}^{-\pi a}, \qquad g_{22}g_{21}-g_{11}g_{12}+s_0^0 g_{11}g_{22}=\mathrm{i}\mathrm{e}^{-\pi a}, \\ g_{11}^2-g_{21}^2-s_0^0 g_{11}g_{21}=\mathrm{i}\mathrm{e}^{-\pi a}s_0^\infty, \quad g_{22}^2-g_{12}^2+s_0^0 g_{12}g_{22}=\mathrm{i}\mathrm{e}^{\pi a}s_1^\infty, \quad g_{11}g_{22}-g_{12}g_{21}=1. \end{gathered} \tag{33}$$

These equations are equivalent to one of the following (three) systems:

**(1)** $g_{11}g_{22}\neq 0 \;\Rightarrow$

$$s_0^\infty = -\frac{(g_{21}+\mathrm{i}g_{11}\mathrm{e}^{\pi a})}{g_{22}}, \qquad s_1^\infty = \frac{(g_{12}-\mathrm{i}g_{22}\mathrm{e}^{-\pi a})}{g_{11}}, \qquad s_0^0 = \frac{\mathrm{i}\mathrm{e}^{-\pi a}}{g_{11}g_{22}}+\frac{g_{12}}{g_{22}}-\frac{g_{21}}{g_{11}};$$

**(2)** $g_{11}=0 \Rightarrow g_{22}\neq 0$, parameters $=(s_0^0, g_{22})$,

$$g_{12}=\mathrm{i}g_{22}\mathrm{e}^{\pi a}, \quad g_{21}=\frac{\mathrm{i}\mathrm{e}^{-\pi a}}{g_{22}}, \quad s_0^\infty=-\frac{\mathrm{i}\mathrm{e}^{-\pi a}}{g_{22}^2}, \quad s_1^\infty=-\mathrm{i}g_{22}^2\left(1+\mathrm{e}^{2\pi a}+\mathrm{i}s_0^0\mathrm{e}^{\pi a}\right)\mathrm{e}^{-\pi a};$$

**(3)** $g_{11}\neq 0 \Rightarrow g_{22}=0$, parameters $=(s_0^0, g_{11})$,

$$g_{12}=-\frac{\mathrm{i}\mathrm{e}^{-\pi a}}{g_{11}}, \quad g_{21}=-\mathrm{i}\mathrm{e}^{\pi a}g_{11}, \quad s_1^\infty=-\frac{\mathrm{i}\mathrm{e}^{-3\pi a}}{g_{11}^2}, \quad s_0^\infty=-\mathrm{i}g_{11}^2\left(1+\mathrm{e}^{2\pi a}+\mathrm{i}s_0^0\mathrm{e}^{\pi a}\right)\mathrm{e}^{\pi a}.$$

**Remark 2.1.** In view of the rescalings mentioned in Remark 1.2, one can renormalise the canonical solutions, namely, $\tau\to\tau/\sqrt{\varepsilon b}$, $\mu\to\mu(\varepsilon b)^{1/4}$, and $Y_k^\infty \to Y_k^\infty \exp\left(-\frac{1}{4}(a-\frac{\mathrm{i}}{2})\ln(\varepsilon b)\sigma_3\right)$, such that the monodromy data will be independent of the parameters $\varepsilon, b$. Thus, the dependence of the solution, $u(\tau)$, on the parameter (product) $\varepsilon b$ is similar to its dependence on $\tau$; in particular, $u(\tau)$ has singular points at $\varepsilon b=0$ and $\infty$. ■

**Proposition 2.3.** *Define the functions*

$$\mathcal{H}_0(\tau) := \operatorname{tr}\left(\tfrac{\mathrm{i}\sqrt{\varepsilon b}}{2\sqrt{\tau}}\mathcal{Z}_1\sigma_3\right) \qquad \text{and} \qquad \mathcal{H}_\infty(\tau) := \operatorname{tr}\left(\mathrm{i}\left(2\Psi^{(2)}-(\Psi^{(1)})^2\right)\sigma_3\right), \tag{34}$$

*where* $\Psi^{(1)}$, $\Psi^{(2)}$, *and* $\mathcal{Z}_1$ *are the coefficients of the asymptotic expansions* (15) *and* (16). *Let*

$$\mathcal{H}(\tau) := \mathcal{H}_0(\tau)+\mathcal{H}_\infty(\tau). \tag{35}$$

*Then*

$$\mathcal{H}_0(\tau)-\mathcal{H}_\infty(\tau) = -\tfrac{1}{2\tau}\left(a-\tfrac{\mathrm{i}}{2}\right)^2, \tag{36}$$

$$\begin{aligned} \mathcal{H}(\tau) &= \frac{\left(a\mathrm{i}+\frac{1}{2}+\frac{2\tau AD}{\sqrt{-AB}}\right)^2}{2\tau}+4\tau\sqrt{-AB}-\frac{\mathrm{i}\varepsilon bD}{B}+2\tau CD+\frac{AD}{\sqrt{-AB}} \\ &= \left(a-\tfrac{\mathrm{i}}{2}\right)\frac{b}{u(\tau)}+\frac{1}{2\tau}\left(a-\tfrac{\mathrm{i}}{2}\right)^2+\frac{\tau}{4u^2(\tau)}\left((u'(\tau))^2+b^2\right)+4\varepsilon u(\tau). \end{aligned} \tag{37}$$

*Proof.* Using the explicit expressions for $\Psi^{(1)}$, $\Psi^{(2)}$, and $\mathcal{Z}_1$ given in Equations (17) and (18), one obtains the first equation of (37) and

$$\mathcal{H}_0(\tau)-\mathcal{H}_\infty(\tau)=\frac{\left(a\mathrm{i}+\frac{1}{2}+\frac{2\tau AD}{\sqrt{-AB}}\right)^2}{2\tau}-\frac{\mathrm{i}\varepsilon bD}{B}-2\tau CD-\frac{AD}{\sqrt{-AB}}.$$

Now, from the formulae for $A(\tau)$, $B(\tau)$, $C(\tau)$, and $D(\tau)$ in terms of $u(\tau)$ and $\varphi(\tau)$ given in Proposition 1.2, one arrives at Equation (36) and the second of (37). □

**Remark 2.2.** Note that $\mathcal{H}_{\varepsilon_1}(p(\tau),u(\tau);\tau)|_{\varepsilon_1=-1}=\mathcal{H}(\tau)$, where $\mathcal{H}_{\varepsilon_1}(p,q;\tau)$ is defined in Equation (7), $\mathcal{H}(\tau)$ is given by the second of (37), and $p(\tau)$ is calculated from the first equation of (11) with $q=u(\tau)$. ■

An important object in the theory of Painlevé equations is the $\boldsymbol{\tau}$-function [12, 13, 30]. In this work, it is defined as follows:

$$\mathcal{H}(\tau):=\partial_\tau \ln(\boldsymbol{\tau}(\tau)).$$

## 3 Summary of Results

In order to present our asymptotic results, it is convenient to introduce the following auxiliary mapping $\mathcal{F}_{\varepsilon_1,\varepsilon_2}\colon \mathcal{M}\to\mathcal{M}$, $(a,s_0^0,s_0^\infty,s_1^\infty,g_{11},g_{12},g_{21},g_{22})\to((-1)^{\varepsilon_2}a,s_0^0,s_0^\infty(\varepsilon_1,\varepsilon_2),s_1^\infty(\varepsilon_1,\varepsilon_2),g_{11}(\varepsilon_1,\varepsilon_2),g_{12}(\varepsilon_1,\varepsilon_2),g_{21}(\varepsilon_1,\varepsilon_2),g_{22}(\varepsilon_1,\varepsilon_2))$, $\varepsilon_1,\varepsilon_2=0,\pm1$. Define:

(1) $\mathcal{F}_{0,0}$ as the identity mapping: $s_0^\infty(0,0)=s_0^\infty$, $s_1^\infty(0,0)=s_1^\infty$, and $g_{ij}(0,0)=g_{ij}$, $i,j=1,2$;

(2) $\mathcal{F}_{0,-1}$ as: $s_0^\infty(0,-1)=s_1^\infty\mathrm{e}^{-\pi a}$, $s_1^\infty(0,-1)=s_0^\infty\mathrm{e}^{-\pi a}$, $g_{11}(0,-1)=-g_{22}\mathrm{e}^{-\frac{\pi a}{2}}$, $g_{12}(0,-1)=-(g_{21}+s_0^\infty g_{22})\mathrm{e}^{\frac{\pi a}{2}}$, $g_{21}(0,-1)=-(g_{12}-s_0^0 g_{22})\mathrm{e}^{-\frac{\pi a}{2}}$, and $g_{22}(0,-1)=-(g_{11}-s_0^0g_{21}+(g_{12}-s_0^0g_{22})s_0^\infty)\mathrm{e}^{\frac{\pi a}{2}}$;

(3) $\mathcal{F}_{0,1}$ as: $s_0^\infty(0,1)=s_1^\infty\mathrm{e}^{-\pi a}$, $s_1^\infty(0,1)=s_0^\infty\mathrm{e}^{-\pi a}$, $g_{11}(0,1)=-\mathrm{i}g_{12}\mathrm{e}^{-\frac{\pi a}{2}}$, $g_{12}(0,1)=-\mathrm{i}(g_{11}+s_0^\infty g_{12})\mathrm{e}^{\frac{\pi a}{2}}$, $g_{21}(0,1)=-\mathrm{i}g_{22}\mathrm{e}^{-\frac{\pi a}{2}}$, and $g_{22}(0,1)=-\mathrm{i}(g_{21}+s_0^\infty g_{22})\mathrm{e}^{\frac{\pi a}{2}}$;

(4) $\mathcal{F}_{-1,0}$ as: $s_0^\infty(-1,0)=-s_0^\infty\mathrm{e}^{-\pi a}$, $s_1^\infty(-1,0)=-s_1^\infty\mathrm{e}^{\pi a}$, $g_{11}(-1,0)=g_{21}\mathrm{e}^{-\frac{\pi a}{2}}$, $g_{12}(-1,0)=-g_{22}\mathrm{e}^{\frac{\pi a}{2}}$, $g_{21}(-1,0)=(g_{11}-s_0^0g_{21})\mathrm{e}^{-\frac{\pi a}{2}}$, and $g_{22}(-1,0)=-(g_{12}-s_0^0g_{22})\mathrm{e}^{\frac{\pi a}{2}}$;

(5) $\mathcal{F}_{-1,-1}$ as: $s_0^\infty(-1,-1)=-s_1^\infty$, $s_1^\infty(-1,-1)=-s_0^\infty\mathrm{e}^{-2\pi a}$, $g_{11}(-1,-1)=g_{12}-s_0^0g_{22}$, $g_{12}(-1,-1)=-g_{11}+s_0^0g_{21}-(g_{12}-s_0^0g_{22})s_0^\infty$, $g_{21}(-1,-1)=g_{22}-(g_{12}-s_0^0g_{22})s_0^0$, and $g_{22}(-1,-1)=-g_{21}+(g_{11}-s_0^0g_{21})s_0^0-(g_{22}-(g_{12}-s_0^0g_{22})s_0^0)s_0^\infty$;

(6) $\mathcal{F}_{-1,1}$ as: $s_0^\infty(-1,1)=-s_1^\infty$, $s_1^\infty(-1,1)=-s_0^\infty\mathrm{e}^{-2\pi a}$, $g_{11}(-1,1)=\mathrm{i}g_{22}$, $g_{12}(-1,1)=-\mathrm{i}(g_{21}+s_0^\infty g_{22})$, $g_{21}(-1,1)=\mathrm{i}(g_{12}-s_0^0g_{22})$, and $g_{22}(-1,1)=-\mathrm{i}(g_{11}-s_0^0g_{21}+(g_{12}-s_0^0g_{22})s_0^\infty)$;

(7) $\mathcal{F}_{1,0}$ as: $s_0^\infty(1,0)=-s_0^\infty\mathrm{e}^{\pi a}$, $s_1^\infty(1,0)=-s_1^\infty\mathrm{e}^{-\pi a}$, $g_{11}(1,0)=(g_{21}+s_0^0g_{11})\mathrm{e}^{\frac{\pi a}{2}}$, $g_{12}(1,0)=-(g_{22}+s_0^0g_{12})\mathrm{e}^{-\frac{\pi a}{2}}$, $g_{21}(1,0)=g_{11}\mathrm{e}^{\frac{\pi a}{2}}$, and $g_{22}(1,0)=-g_{12}\mathrm{e}^{-\frac{\pi a}{2}}$;

(8) $\mathcal{F}_{1,-1}$ as: $s_0^\infty(1,-1)=-s_1^\infty\mathrm{e}^{-2\pi a}$, $s_1^\infty(1,-1)=-s_0^\infty$, $g_{11}(1,-1)=g_{12}\mathrm{e}^{-\pi a}$, $g_{12}(1,-1)=-(g_{11}+s_0^\infty g_{12})\mathrm{e}^{\pi a}$, $g_{21}(1,-1)=g_{22}\mathrm{e}^{-\pi a}$, and $g_{22}(1,-1)=-(g_{21}+s_0^\infty g_{22})\mathrm{e}^{\pi a}$; and

(9) $\mathcal{F}_{1,1}$ as: $s_0^\infty(1,1)=-s_1^\infty\mathrm{e}^{-2\pi a}$, $s_1^\infty(1,1)=-s_0^\infty$, $g_{11}(1,1)=\mathrm{i}(g_{22}+s_0^0g_{12})\mathrm{e}^{-\pi a}$, $g_{12}(1,1)=-\mathrm{i}(g_{21}+s_0^0g_{11}+(g_{22}+s_0^0g_{12})s_0^\infty)\mathrm{e}^{\pi a}$, and $g_{22}(1,1)=-\mathrm{i}(g_{11}+s_0^\infty g_{12})\mathrm{e}^{\pi a}$.

**Remark 3.1.** Throughout this work, $\epsilon$ and $\delta$ (assumed sufficiently small), with and without subscripts, superscripts, etc., denote positive real numbers, and the context(s) in which they appear should make clear the parameter(s), if any, on which they depend. The roots of positive quantities are assumed positive, whilst the branches of the roots of complex quantities can be taken arbitrarily, unless stated otherwise. For negative values of a real variable $\varkappa$ ($\varkappa=\tau$ or $\varepsilon b$, say), $\varkappa^{1/3}:=-|\varkappa|^{1/3}$. ■

**Theorem 3.1.** *Let $\varepsilon_1, \varepsilon_2 = 0, \pm 1$, $\varepsilon b = |\varepsilon b| \mathrm{e}^{\mathrm{i}\pi\varepsilon_2}$, and $u(\tau)$ be a solution of Equation* (1) *corresponding to the monodromy data $(a, s_0^0, s_0^\infty, s_1^\infty, g_{11}, g_{12}, g_{21}, g_{22})$. Suppose that*

$$g_{11}(\varepsilon_1,\varepsilon_2)g_{12}(\varepsilon_1,\varepsilon_2)g_{21}(\varepsilon_1,\varepsilon_2)g_{22}(\varepsilon_1,\varepsilon_2)\neq 0, \quad \left|\mathrm{Re}\left(\tfrac{\mathrm{i}}{2\pi}\ln(g_{11}(\varepsilon_1,\varepsilon_2)g_{22}(\varepsilon_1,\varepsilon_2))\right)\right| < \frac{1}{6}. \tag{38}$$

*Then $\exists\, \delta > 0$ such that $u(\tau)$ has the asymptotic expansion*

$$\begin{aligned} u(\tau) \underset{\tau\to\infty\mathrm{e}^{\mathrm{i}\pi\varepsilon_1}}{=} & \frac{(-1)^{\varepsilon_1}\varepsilon\sqrt{|\varepsilon b|}}{3^{1/4}}\left(\sqrt{\frac{\vartheta(\tau)}{12}} + \sqrt{\widetilde{\nu}(\varepsilon_1,\varepsilon_2)+1}\,\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}\cosh\!\Big(\mathrm{i}\vartheta(\tau) + (\widetilde{\nu}(\varepsilon_1,\varepsilon_2)+1)\ln\vartheta(\tau)\right. \\ & \left. +\; z(\varepsilon_1,\varepsilon_2) + o\left(\tau^{-\delta}\right)\Big)\right), \end{aligned} \tag{39}$$

*where*

$$\vartheta(\tau) := 3\sqrt{3}\,|\varepsilon b|^{1/3}|\tau|^{2/3}, \qquad \widetilde{\nu}(\varepsilon_1,\varepsilon_2)+1 := \frac{\mathrm{i}}{2\pi}\ln(g_{11}(\varepsilon_1,\varepsilon_2)g_{22}(\varepsilon_1,\varepsilon_2)),$$

$$\begin{aligned} z(\varepsilon_1,\varepsilon_2) := & \frac{1}{2}\ln(2\pi) - \frac{\pi\mathrm{i}}{2} - \frac{3\pi\mathrm{i}}{2}(\widetilde{\nu}(\varepsilon_1,\varepsilon_2)+1) + (-1)^{\varepsilon_2}\mathrm{i}a\ln\left(2+\sqrt{3}\right) + (\widetilde{\nu}(\varepsilon_1,\varepsilon_2)+1)\ln 12 \\ & - \ln\left(\omega(\varepsilon_1,\varepsilon_2)\sqrt{\widetilde{\nu}(\varepsilon_1,\varepsilon_2)+1}\,\Gamma(\widetilde{\nu}(\varepsilon_1,\varepsilon_2)+1)\right), \end{aligned}$$

*with*

$$\omega(\varepsilon_1,\varepsilon_2) := \frac{g_{12}(\varepsilon_1,\varepsilon_2)}{g_{22}(\varepsilon_1,\varepsilon_2)},$$

*and $\Gamma(\cdot)$ is the gamma function* [31].

*Let $\mathcal{H}(\tau)$ be the Hamiltonian function defined in Equation* (35) *corresponding to the function $u(\tau)$ given above. Then*

$$\begin{aligned} \mathcal{H}(\tau) \underset{\tau\to\infty\mathrm{e}^{\mathrm{i}\pi\varepsilon_1}}{=} & 3(\varepsilon b)^{2/3}\tau^{1/3} + 2|\varepsilon b|^{1/3}\tau^{-1/3}\left((a-(-1)^{\varepsilon_2}\mathrm{i}/2) - 2\sqrt{3}\,\mathrm{i}(\widetilde{\nu}(\varepsilon_1,\varepsilon_2)+1) + o\left(\tau^{-\delta}\right)\right) \\ & + \frac{(a-(-1)^{\varepsilon_2}\mathrm{i}/2)^2}{2\tau}. \end{aligned} \tag{40}$$

**Remark 3.2.** If conditions (38) are valid for two distinct pairs of values of $(\varepsilon_1,\varepsilon_2)$, $(\varepsilon_1^\flat,\varepsilon_2^\flat)$ and $(\varepsilon_1^\natural,\varepsilon_2^\natural)$, say, then Theorem 3.1 yields connection formulae for asymptotics of $u(\tau)$ as $\tau\to\infty\mathrm{e}^{\mathrm{i}\pi\varepsilon_1^\flat}$ and $\tau\to\infty\mathrm{e}^{\mathrm{i}\pi\varepsilon_1^\natural}$; for example, the connection formulae in terms of the parameters $\omega(\pm1,0)$ and $\widetilde{\nu}(\pm1,0)$ read:

$$\begin{gathered} \omega(1,0) = \omega + \left(\mathrm{i}\mathrm{e}^{-\pi a} + \frac{1}{\omega}\right)\mathrm{e}^{2\pi\mathrm{i}(\widetilde{\nu}+1)}, \\ \mathrm{e}^{-2\pi\mathrm{i}(\widetilde{\nu}(1,0)+1)} = -\omega\left(\omega\mathrm{e}^{-2\pi\mathrm{i}(\widetilde{\nu}+1)} + \mathrm{i}\mathrm{e}^{-\pi a}\right), \\ \omega(-1,0) = \frac{\omega\mathrm{e}^{2\pi\mathrm{i}(\widetilde{\nu}+1)}}{\mathrm{e}^{2\pi\mathrm{i}(\widetilde{\nu}+1)} - 1 - \mathrm{i}\omega\mathrm{e}^{-\pi a}}, \\ \mathrm{e}^{-2\pi\mathrm{i}(\widetilde{\nu}(-1,0)+1)} = \frac{1}{\omega^2}\left(1-\mathrm{e}^{2\pi\mathrm{i}(\widetilde{\nu}+1)}\right)\left(1-\mathrm{e}^{-2\pi\mathrm{i}(\widetilde{\nu}+1)} + \mathrm{i}\omega\mathrm{e}^{-\pi a}\right), \end{gathered}$$

where $\omega := \omega(0,0)$, and $\widetilde{\nu} := \widetilde{\nu}(0,0)$. ■

**Theorem 3.2.** *Let $\varepsilon_1, \varepsilon_2 = 0, \pm 1$, $\varepsilon b = |\varepsilon b| \mathrm{e}^{\mathrm{i}\pi\varepsilon_2}$, and $u(\tau)$ be a solution of Equation* (1) *corresponding to the monodromy data $(a, s_0^0, s_0^\infty, s_1^\infty, g_{11}, g_{12}, g_{21}, g_{22})$. Suppose that*

$$g_{21}(\varepsilon_1,\varepsilon_2) = 0, \qquad g_{11}(\varepsilon_1,\varepsilon_2)g_{22}(\varepsilon_1,\varepsilon_2) = 1.$$

*Then $\exists\,\delta>0$ such that $u(\tau)$ has the asymptotic expansion*

$$u(\tau) \underset{\tau\to\infty e^{i\pi\varepsilon_1}}{=} \frac{\varepsilon(\varepsilon b)^{2/3}}{2}\tau^{1/3}+\frac{(-1)^{\varepsilon_1}\varepsilon\sqrt{|\varepsilon b|}\,(s_0^0-\mathrm{i}e^{(-1)^{\varepsilon_2+1}\pi a})}{2^{3/2}3^{1/4}\sqrt{\pi}}\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)^{(-1)^{\varepsilon_2}\mathrm{i}a}$$
$$\times\,\exp\Big(-\mathrm{i}\Big(3\sqrt{3}\,|\varepsilon b|^{1/3}|\tau|^{2/3}-\frac{\pi}{4}\Big)\Big)\Big(1+o\Big(\tau^{-\delta}\Big)\Big). \tag{41}$$

**Theorem 3.3.** *Let $\varepsilon_1,\varepsilon_2=0,\pm1$, $\varepsilon b=|\varepsilon b|e^{i\pi\varepsilon_2}$, and $u(\tau)$ be a solution of Equation* (1) *corresponding to the monodromy data $(a,s_0^0,s_0^\infty,s_1^\infty,g_{11},g_{12},g_{21},g_{22})$. Suppose that*

$$g_{12}(\varepsilon_1,\varepsilon_2)=0,\qquad g_{11}(\varepsilon_1,\varepsilon_2)g_{22}(\varepsilon_1,\varepsilon_2)=1.$$

*Then $\exists\,\delta>0$ such that $u(\tau)$ has the asymptotic expansion*

$$u(\tau) \underset{\tau\to\infty e^{i\pi\varepsilon_1}}{=} \frac{\varepsilon(\varepsilon b)^{2/3}}{2}\tau^{1/3}+\frac{(-1)^{\varepsilon_1}\varepsilon\sqrt{|\varepsilon b|}\,(s_0^0-\mathrm{i}e^{(-1)^{\varepsilon_2+1}\pi a})}{2^{3/2}3^{1/4}\sqrt{\pi}}\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)^{(-1)^{\varepsilon_2}\mathrm{i}a}$$
$$\times\,\exp\Big(\mathrm{i}\Big(3\sqrt{3}\,|\varepsilon b|^{1/3}|\tau|^{2/3}+\frac{3\pi}{4}\Big)\Big)\Big(1+o\Big(\tau^{-\delta}\Big)\Big). \tag{42}$$

**Remark 3.3.** The function $\mathcal{H}(\tau)$ also has asymptotics (40) for the conditions on the monodromy data given in Theorems 3.2 and 3.3. ■

**Theorem 3.4.** *Let $\varepsilon_1,\varepsilon_2=0,\pm1$, $\varepsilon b=|\varepsilon b|e^{i\pi\varepsilon_2}$, and $u(\tau)$ be a solution of Equation* (1) *corresponding to the monodromy data $(a,s_0^0,s_0^\infty,s_1^\infty,g_{11},g_{12},g_{21},g_{22})$. Suppose that*

$$|\mathrm{Im}(a)|<1,\qquad g_{11}(\varepsilon_1,\varepsilon_2)g_{22}(\varepsilon_1,\varepsilon_2)\neq0,\qquad \rho\neq0,\qquad |\mathrm{Re}(\rho)|<\frac{1}{2}, \tag{43}$$

*where*

$$\cos(2\pi\rho):=-\frac{\mathrm{i}s_0^0}{2}=\cosh(\pi a)+\frac{1}{2}s_0^\infty s_1^\infty e^{\pi a}. \tag{44}$$

*Then $\exists\,\delta>0$ such that $u(\tau)$ has the asymptotic expansion*

$$u(\tau) \underset{\tau\to0e^{i\pi\varepsilon_1}}{=} \frac{(-1)^{\varepsilon_2}\tau b}{16\pi}\exp\Big((-1)^{\varepsilon_2}\frac{\pi a}{2}\Big)\Big(\mathfrak{p}((-1)^{\varepsilon_2}a,\rho)\chi_1(\vec{g}(\varepsilon_1,\varepsilon_2);\rho)|\tau|^{2\rho}+\mathfrak{p}((-1)^{\varepsilon_2}a,-\rho)$$
$$\times\;\chi_1(\vec{g}(\varepsilon_1,\varepsilon_2);-\rho)|\tau|^{-2\rho}\Big)\Big(\mathfrak{p}((-1)^{\varepsilon_2+1}a,\rho)e^{-i\pi\rho}\chi_2(\vec{g}(\varepsilon_1,\varepsilon_2);\rho)|\tau|^{2\rho}$$
$$+\;\mathfrak{p}((-1)^{\varepsilon_2+1}a,-\rho)e^{i\pi\rho}\chi_2(\vec{g}(\varepsilon_1,\varepsilon_2);-\rho)|\tau|^{-2\rho}\Big)\Big(1+\mathcal{O}\Big(\tau^{\delta}\Big)\Big), \tag{45}$$

*where*

$$\mathfrak{p}(z_1,z_2):=\left(\frac{|\varepsilon b|}{32}e^{\frac{i\pi}{2}}\right)^{z_2}\left(\frac{\Gamma(\frac{1}{2}-z_2)}{\Gamma(1+z_2)}\right)^2\frac{\Gamma(1+z_2+\frac{iz_1}{2})}{\tan(\pi z_2)}, \tag{46}$$

$$\begin{aligned}\chi_1(\vec{g}(\varepsilon_1,\varepsilon_2);z_3)&:=g_{11}(\varepsilon_1,\varepsilon_2)e^{i\pi z_3}e^{\frac{i\pi}{4}}+g_{21}(\varepsilon_1,\varepsilon_2)e^{-i\pi z_3}e^{-\frac{i\pi}{4}},\\ \chi_2(\vec{g}(\varepsilon_1,\varepsilon_2);z_4)&:=g_{12}(\varepsilon_1,\varepsilon_2)e^{i\pi z_4}e^{\frac{i\pi}{4}}+g_{22}(\varepsilon_1,\varepsilon_2)e^{-i\pi z_4}e^{-\frac{i\pi}{4}}.\end{aligned} \tag{47}$$

*Let $\mathcal{H}(\tau)$ be the Hamiltonian function defined in Equation* (35) *corresponding to the function $u(\tau)$ given above. Then*

$$\mathcal{H}(\tau) \underset{\tau\to0e^{i\pi\varepsilon_1}}{=} \frac{2\rho}{\tau}\frac{\big(\mathfrak{p}((-1)^{\varepsilon_2}a,\rho)\chi_1(\vec{g}(\varepsilon_1,\varepsilon_2);\rho)|\tau|^{2\rho}-\mathfrak{p}((-1)^{\varepsilon_2}a,-\rho)\chi_1(\vec{g}(\varepsilon_1,\varepsilon_2);-\rho)|\tau|^{-2\rho}\big)}{\big(\mathfrak{p}((-1)^{\varepsilon_2},\rho)\chi_1(\vec{g}(\varepsilon_1,\varepsilon_2);\rho)|\tau|^{2\rho}+\mathfrak{p}((-1)^{\varepsilon_2}a,-\rho)\chi_1(\vec{g}(\varepsilon_1,\varepsilon_2);-\rho)|\tau|^{-2\rho}\big)}$$
$$+\,\frac{1}{2\tau}\Big(a(a-(-1)^{\varepsilon_2}\mathrm{i})+\frac{1}{4}+8\rho^2\Big)+o\Big(\frac{1}{\tau}\Big). \tag{48}$$

**Theorem 3.5.** *Let $\varepsilon_1,\varepsilon_2=0,\pm1$, $\varepsilon b=|\varepsilon b|\mathrm{e}^{\mathrm{i}\pi\varepsilon_2}$, and $u(\tau)$ be a solution of Equation* (1) *corresponding to the monodromy data $(a,s_0^0,s_0^\infty,s_1^\infty,g_{11},g_{12},g_{21},g_{22})$. Suppose that*

$$|\mathrm{Im}(a)|<1, \qquad g_{11}(\varepsilon_1,\varepsilon_2)g_{22}(\varepsilon_1,\varepsilon_2)\neq 0, \qquad s_0^0=2\mathrm{i}. \tag{49}$$

*Then $\exists\ \delta>0$ such that $u(\tau)$ has the asymptotic expansion*

$$\begin{aligned} u(\tau) \underset{\tau\to 0\mathrm{e}^{\mathrm{i}\pi\varepsilon_1}}{=} & \frac{(-1)^{\varepsilon_2}\tau b\exp((-1)^{\varepsilon_2}\frac{\pi a}{2})}{2a\sinh(\frac{\pi a}{2})}\bigg(\chi_1(\vec{g}(\varepsilon_1,\varepsilon_2);0)\Big(1-\tfrac{(-1)^{\varepsilon_2}\mathrm{i}a}{2}Q((-1)^{\varepsilon_2}a)\Big)+\frac{(-1)^{\varepsilon_2}\pi a}{4}\\ &\times\ (g_{21}(\varepsilon_1,\varepsilon_2)\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}-3g_{11}(\varepsilon_1,\varepsilon_2)\mathrm{e}^{\frac{\mathrm{i}\pi}{4}})+(-1)^{\varepsilon_2}\mathrm{i}a\chi_1(\vec{g}(\varepsilon_1,\varepsilon_2);0)\ln|\tau|\bigg)\\ &\times\ \bigg(\chi_2(\vec{g}(\varepsilon_1,\varepsilon_2);0)\Big(1+\tfrac{(-1)^{\varepsilon_2}\mathrm{i}a}{2}Q((-1)^{\varepsilon_2+1}a)\Big)+\frac{(-1)^{\varepsilon_2}\pi a}{4}(g_{12}(\varepsilon_1,\varepsilon_2)\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\\ &-\ 3g_{22}(\varepsilon_1,\varepsilon_2)\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}})-(-1)^{\varepsilon_2}\mathrm{i}a\chi_2(\vec{g}(\varepsilon_1,\varepsilon_2);0)\ln|\tau|\bigg)\Big(1+\mathcal{O}\Big(\tau^{\delta}\Big)\Big), \end{aligned} \tag{50}$$

*where $\chi_j(\vec{g}(\varepsilon_1,\varepsilon_2);\boldsymbol{\cdot})$, $j=1,2$, are defined in Theorem* 3.4*, Equations* (47)*,*

$$Q(z):=4\psi(1)-\psi(\mathrm{i}z/2)+\ln 2-\ln(|\varepsilon b|), \tag{51}$$

*$\psi(x):=\frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x)$ is the psi function, and $\psi(1)=-0.57721566490\ldots$* [31].

*Let $\mathcal{H}(\tau)$ be the Hamiltonian function defined in Equation* (35) *corresponding to the function $u(\tau)$ given above. Then*

$$\mathcal{H}(\tau)\underset{\tau\to 0\mathrm{e}^{\mathrm{i}\pi\varepsilon_1}}{=}\frac{1}{2\tau}\left(a(a-(-1)^{\varepsilon_2}\mathrm{i})+\frac{1}{4}\right)+\frac{b_2(\varepsilon_1,\varepsilon_2)}{\tau(a_2(\varepsilon_1,\varepsilon_2)+b_2(\varepsilon_1,\varepsilon_2)\ln|\tau|)}+o\left(\frac{1}{\tau}\right), \tag{52}$$

*where*

$$\begin{aligned} a_2(\varepsilon_1,\varepsilon_2) :=&\ \chi_1(\vec{g}(\varepsilon_1,\varepsilon_2);0)\Big(1-\tfrac{(-1)^{\varepsilon_2}\mathrm{i}a}{2}Q((-1)^{\varepsilon_2}a)\Big)+\frac{(-1)^{\varepsilon_2}\pi a}{4}(g_{21}(\varepsilon_1,\varepsilon_2)\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}\\ &-3g_{11}(\varepsilon_1,\varepsilon_2)\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}), \end{aligned}$$

$$b_2(\varepsilon_1,\varepsilon_2):=(-1)^{\varepsilon_2}\mathrm{i}a\chi_1(\vec{g}(\varepsilon_1,\varepsilon_2);0).$$

**Conjecture 3.1.** *For the conditions stated in Theorem* 3.4*,*

$$\begin{aligned} \boldsymbol{\tau}(\tau)\underset{\tau\to 0\mathrm{e}^{\mathrm{i}\pi\varepsilon_1}}{=}&\ \mathrm{const.}\,\tau^{\frac{1}{2}\left(a(a-(-1)^{\varepsilon_2}\mathrm{i})+\frac{1}{4}+8\rho^2\right)}\big(\mathfrak{p}((-1)^{\varepsilon_2}a,\rho)\chi_1(\vec{g}(\varepsilon_1,\varepsilon_2);\rho)|\tau|^{2\rho}\\ &+\ \mathfrak{p}((-1)^{\varepsilon_2}a,-\rho)\chi_1(\vec{g}(\varepsilon_1,\varepsilon_2);-\rho)|\tau|^{-2\rho}\big)\Big(1+o\Big(\tau^{\delta}\Big)\Big). \end{aligned} \tag{53}$$

**Conjecture 3.2.** *For the conditions stated in Theorem* 3.5*,*

$$\boldsymbol{\tau}(\tau)\underset{\tau\to 0\mathrm{e}^{\mathrm{i}\pi\varepsilon_1}}{=}\mathrm{const.}\,\tau^{\frac{1}{2}\left(a(a-(-1)^{\varepsilon_2}\mathrm{i})+\frac{1}{4}\right)}(a_2(\varepsilon_1,\varepsilon_2)+b_2(\varepsilon_1,\varepsilon_2)\ln|\tau|)\Big(1+o\Big(\tau^{\delta}\Big)\Big). \tag{54}$$

**Remark 3.4.** Conjectures 3.1 and 3.2 can be proved under the assumption that the error terms in Equations (48) and (52) behave as power-like functions under integration. One can also obtain asymptotics for $\boldsymbol{\tau}(\tau)$ as $\tau\to\infty\mathrm{e}^{\mathrm{i}\pi\varepsilon_1}$, $\varepsilon_1=0,\pm1$, up to $\exp(\mathcal{O}(\tau^{2/3}))$, under similar assumptions on the error term in Equation (40). ■

**Remark 3.5.** The asymptotic results as $\tau\to+\infty$ for the functions $A(\tau)$, $B(\tau)$, $C(\tau)$, and $D(\tau)$, which solve System (5), can be found, via Equations (56), in Proposition 4.3.1. The asymptotics as $\tau\to+0$ for these functions for $\rho\neq0$ (resp., $\rho=0$) are given in Proposition 5.5 (resp., Proposition 5.7) via Proposition 5.6. Asymptotics of these functions for negative and (pure) imaginary values of $\tau$ can be obtained by applying the transformations given in Section 6, as was done for $u(\tau)$ (see Subsection 6.2 and the Appendix). ■

# 4 Calculation of the Monodromy Data as $\tau \to +\infty$

In this section the results stated in Theorem 3.1 for $\tau \to +\infty$ and $\varepsilon b > 0$ are derived. The corresponding results stated in Theorems 3.2 and 3.3 can be obtained analogously. The derivation is based on the WKB analysis of the $\mu$-part of System (12):

$$\partial_\mu \Psi = \widetilde{\mathcal{U}}(\mu, \tau)\Psi. \tag{55}$$

## 4.1 WKB Analysis

This subsection is devoted to the WKB analysis of Equation (55) as $\tau \to +\infty$. In order to put Equation (55) into a form suitable for WKB analysis, it is convenient to introduce the notation given in Proposition 4.1.1 below.

**Proposition 4.1.1.** *Let*

$$\begin{gathered} A(\tau) = a(\tau)\tau^{-2/3}, \qquad B(\tau) = b(\tau)\tau^{-2/3}, \qquad C(\tau) = c(\tau)\tau^{-1/3}, \qquad D(\tau) = d(\tau)\tau^{-1/3}, \\ \widetilde{\mu} = \mu\tau^{1/6}, \qquad \widetilde{\Psi}(\widetilde{\mu}) = \tau^{-(1/12)\sigma_3}\Psi(\widetilde{\mu}\tau^{-1/6}). \end{gathered} \tag{56}$$

*Then*

$$\partial_{\widetilde{\mu}}\widetilde{\Psi}(\widetilde{\mu}) = \tau^{2/3}\mathcal{A}(\widetilde{\mu}, \tau)\widetilde{\Psi}(\widetilde{\mu}), \tag{57}$$

*where*

$$\mathcal{A}(\widetilde{\mu}, \tau) := -2\mathrm{i}\widetilde{\mu}\sigma_3 + \begin{pmatrix} 0 & -\frac{4\mathrm{i}\sqrt{-a(\tau)b(\tau)}}{b(\tau)} \\ -2d(\tau) & 0 \end{pmatrix} - \frac{\mathrm{i}r(\tau)(\varepsilon b)^{1/3}}{2\widetilde{\mu}}\sigma_3 + \frac{1}{\widetilde{\mu}^2}\begin{pmatrix} 0 & \frac{\mathrm{i}\varepsilon b}{b(\tau)} \\ \mathrm{i}b(\tau) & 0 \end{pmatrix}, \tag{58}$$

*with*

$$\frac{\mathrm{i}r(\tau)(\varepsilon b)^{1/3}}{2} := \left(a\mathrm{i} + \frac{1}{2}\right)\tau^{-2/3} + \frac{2a(\tau)d(\tau)}{\sqrt{-a(\tau)b(\tau)}}. \tag{59}$$

*Proof.* Equations (57)–(59) are obtained by the direct substitution of Equations (56) into Equation (55), upon taking into account Lemma 2.1. □

The WKB analysis for Equation (57) is carried out under the following conditions:

$$a(\tau)d(\tau) + b(\tau)c(\tau) + a\mathrm{i}\sqrt{-a(\tau)b(\tau)}\,\tau^{-2/3} = -\frac{\mathrm{i}\varepsilon b}{2}, \tag{60}$$

$$r(\tau) = -2 + r_0(\tau)\tau^{-1/3}, \tag{61}$$

$$\sqrt{-a(\tau)b(\tau)} + c(\tau)d(\tau) + \frac{a(\tau)d(\tau)\tau^{-2/3}}{2\sqrt{-a(\tau)b(\tau)}} - \frac{(a - \mathrm{i}/2)^2\tau^{-4/3}}{4} = \frac{3(\varepsilon b)^{2/3}}{4} - h_0(\tau)\tau^{-2/3}, \tag{62}$$

$$\frac{2\sqrt{-a(\tau)b(\tau)}}{(\varepsilon b)^{2/3}} = 1 + u_0(\tau)\tau^{-1/3}, \tag{63}$$

where

$$h_0(\tau) \underset{\tau\to+\infty}{=} \mathcal{O}(1), \quad |r_0(\tau)| \underset{\tau\to+\infty}{\leqslant} \mathcal{O}\left(\tau^{\delta_\diamondsuit}\right), \quad |u_0(\tau)| \underset{\tau\to+\infty}{\leqslant} \mathcal{O}\left(\tau^{\delta_\diamondsuit}\right), \quad 0 < \delta_\diamondsuit < \frac{1}{9}. \tag{64}$$

**Remark 4.1.1.** Equation (60) is the integral of motion for System (5) in terms of $a(\tau)$, $b(\tau)$, $c(\tau)$, and $d(\tau)$. Actually, in conditions (61)–(63) the functions $r_0(\tau)$, $h_0(\tau)$, and $u_0(\tau)$, with the power-like (growth) behaviours given in Equation (64), are introduced. Conditions (61)–(63), although somewhat artificial looking at this stage, will be clarified as they appear in the

following (asymptotic) analysis. It is worth noting that conditions (60)–(63) are self-consistent; in fact, they are equivalent to

$$\frac{a(\tau)d(\tau)}{\sqrt{-a(\tau)b(\tau)}}=-\frac{\mathrm{i}(\varepsilon b)^{1/3}}{2}+\frac{\mathrm{i}(\varepsilon b)^{1/3}r_0(\tau)}{4}\tau^{-1/3}-\frac{\mathrm{i}}{2}\left(a-\frac{\mathrm{i}}{2}\right)\tau^{-2/3}, \tag{65}$$

$$\frac{b(\tau)c(\tau)}{\sqrt{-a(\tau)b(\tau)}}=-\frac{\mathrm{i}(\varepsilon b)^{1/3}}{2}+\mathrm{i}(\varepsilon b)^{1/3}\left(\frac{u_0(\tau)}{1+u_0(\tau)\tau^{-1/3}}-\frac{r_0(\tau)}{4}\right)\tau^{-1/3}-\frac{\mathrm{i}}{2}\left(a+\frac{\mathrm{i}}{2}\right)\tau^{-2/3}, \tag{66}$$

$$\begin{aligned}-h_0(\tau)&=\frac{(\varepsilon b)^{2/3}}{2}\left(\frac{u_0^2(\tau)}{1+u_0(\tau)\tau^{-1/3}}+\frac{r_0(\tau)}{2}\left(\frac{u_0(\tau)}{1+u_0(\tau)\tau^{-1/3}}-\frac{r_0(\tau)}{4}\right)\right)\\&\quad+\frac{(\varepsilon b)^{1/3}}{2}\left(a-\frac{\mathrm{i}}{2}\right)-\frac{(\varepsilon b)^{1/3}(a-\mathrm{i}/2)u_0(\tau)}{2(1+u_0(\tau)\tau^{-1/3})}\tau^{-1/3}. \qquad\blacksquare\end{aligned} \tag{67}$$

In certain domains (see below) of the complex $\widetilde{\mu}$-plane, the leading term of asymptotics (as $\tau\to+\infty$) of a fundamental solution, $\widetilde{\Psi}(\widetilde{\mu})$, of Equation (57) is given by the following WKB formula [29],

$$\widetilde{\Psi}_{\mathrm{WKB}}(\widetilde{\mu})=T(\widetilde{\mu})\exp\left(-\mathrm{i}\tau^{2/3}\sigma_3\int^{\widetilde{\mu}}l(\xi)\,\mathrm{d}\xi-\int^{\widetilde{\mu}}\operatorname{diag}\big((T(\xi))^{-1}\partial_{\xi}T(\xi)\big)\,\mathrm{d}\xi\right), \tag{68}$$

where

$$l(\widetilde{\mu})=l(\widetilde{\mu},\tau):=(\det(\mathcal{A}(\widetilde{\mu})))^{1/2}, \tag{69}$$

and $T(\widetilde{\mu})$ diagonalises $\mathcal{A}(\widetilde{\mu})$, that is, $(T(\widetilde{\mu}))^{-1}\mathcal{A}(\widetilde{\mu})T(\widetilde{\mu})=-\mathrm{i}l(\widetilde{\mu})\sigma_3$ (the $\tau$ dependencies of $\widetilde{\Psi}_{\mathrm{WKB}}(\widetilde{\mu})$, $l(\widetilde{\mu})$, and $T(\widetilde{\mu})$ have been suppressed in order to simplify the notation). It is convenient to choose $T(\widetilde{\mu})$ as follows,

$$T(\widetilde{\mu})=\frac{\mathrm{i}}{\sqrt{2\mathrm{i}l(\widetilde{\mu})(\mathcal{A}_{11}(\widetilde{\mu})-\mathrm{i}l(\widetilde{\mu}))}}(\mathcal{A}(\widetilde{\mu})-\mathrm{i}l(\widetilde{\mu})\sigma_3)\sigma_3, \tag{70}$$

where $\mathcal{A}_{11}(\widetilde{\mu})$ is the corresponding element of matrix $\mathcal{A}(\widetilde{\mu})$. Note that $\det(T(\widetilde{\mu}))=1$ and $T(\widetilde{\mu})\underset{\widetilde{\mu}\to\infty}{=}\mathrm{I}+\mathcal{O}(\digamma(\tau)\widetilde{\mu}^{-1})$; therefore, $\det(\widetilde{\Psi}_{\mathrm{WKB}}(\widetilde{\mu}))=1$ (since $\operatorname{tr}((T(\widetilde{\mu}))^{-1}\partial_{\widetilde{\mu}}T(\widetilde{\mu}))=0$). The domains in the complex $\widetilde{\mu}$-plane where Equation (68) gives the asymptotic approximation of solutions of Equation (57) are defined in terms of the *Stokes graph*. The vertices of the Stokes graph are the singular points of Equation (57), namely, $\widetilde{\mu}=0$ and $\infty$, and the *turning points*, which are the roots of the equation $l^2(\widetilde{\mu})=0$. The edges of the Stokes graph are the *Stokes curves*, defined as $\mathrm{Re}(\int_{\widetilde{\mu}^{\mathrm{TP}}}^{\widetilde{\mu}}l(\xi)\,\mathrm{d}\xi)=0$, where $\widetilde{\mu}^{\mathrm{TP}}$ denotes a turning point. *Canonical domains* are those domains in the complex $\widetilde{\mu}$-plane containing one, and only one, Stokes curve and bounded by two adjacent Stokes curves. Note that the restriction of any branch of $l(\widetilde{\mu})$ to a canonical domain is a single-valued function. In each canonical domain, for any choice of the branch of $l(\widetilde{\mu})$, there exists a fundamental solution of Equation (57) which has asymptotics whose leading term as $\tau\to+\infty$ is given by Equation (68). Recalling the definition of $l(\widetilde{\mu})$ (Equation (69)), one shows that

$$l^2(\widetilde{\mu})=\frac{4}{\widetilde{\mu}^4}\left(\left(\widetilde{\mu}^2-\frac{1}{2}(\varepsilon b)^{1/3}\right)^2\left(\widetilde{\mu}^2+(\varepsilon b)^{1/3}\right)+\left(\widetilde{\mu}^4\left(a-\frac{\mathrm{i}}{2}\right)+\widetilde{\mu}^2h_0(\tau)\right)\tau^{-2/3}\right). \tag{71}$$

It follows from Equation (71) that there are six turning points: two turning points coalesce (as $\tau\to+\infty$) at $\frac{(\varepsilon b)^{1/6}}{\sqrt{2}}+\mathcal{O}(\tau^{-1/3})$, another pair coalesce (as $\tau\to+\infty$) at $-\frac{(\varepsilon b)^{1/6}}{\sqrt{2}}+\mathcal{O}(\tau^{-1/3})$, and the remaining two turning points approach (as $\tau\to+\infty$), respectively, $\pm\mathrm{i}(\varepsilon b)^{1/6}+\mathcal{O}(\tau^{-4/9})$. Denote by $\widetilde{\mu}_1$ the turning point in the first quadrant of the complex $\widetilde{\mu}$-plane which approaches $\frac{(\varepsilon b)^{1/6}}{\sqrt{2}}+\mathcal{O}(\tau^{-1/3})$ as $\tau\to+\infty$, and by $\widetilde{\mu}_2$ the pure imaginary turning point which approaches $\mathrm{i}(\varepsilon b)^{1/6}+\mathcal{O}(\tau^{-4/9})$ as $\tau\to+\infty$. Denote by $\mathcal{G}_1$ the part of the Stokes graph in quadrant one of the complex $\widetilde{\mu}$-plane which consists of the vertices $0$, $\infty$, $\widetilde{\mu}_1$, and $\widetilde{\mu}_2$, and the edges $(+\mathrm{i}\infty,\widetilde{\mu}_2)$, $(0,\widetilde{\mu}_1)$, $(\widetilde{\mu}_2,\widetilde{\mu}_1)$, and $(\widetilde{\mu}_1,+\infty)$: the complete Stokes graph is the union of the mirror images of $\mathcal{G}_1$ with respect to the real and imaginary axes of the complex $\widetilde{\mu}$-plane.

**Proposition 4.1.2.**

$$\int_{\widetilde{\mu}^{\mathrm{TP}}}^{\widetilde{\mu}} l(\xi)\,\mathrm{d}\xi \underset{\tau\to+\infty}{=} \left(\left(\xi+\frac{(\varepsilon b)^{1/3}}{\xi}\right)\sqrt{\xi^2+(\varepsilon b)^{1/3}}+\tau^{-2/3}\left(\left(a-\frac{\mathrm{i}}{2}\right)\ln\left(2\sqrt{\xi^2+(\varepsilon b)^{1/3}}+2\xi\right)\right.\right.$$
$$+\frac{1}{2\sqrt{3}}\left(\left(a-\frac{\mathrm{i}}{2}\right)+\frac{2h_0(\tau)}{(\varepsilon b)^{1/3}}\right)\ln\left(\frac{(-\xi+\sqrt{2}\,(\varepsilon b)^{1/6}+\sqrt{3}\,\sqrt{\xi^2+(\varepsilon b)^{1/3}}\,)(\xi-\frac{(\varepsilon b)^{1/6}}{\sqrt{2}})}{(\xi+\sqrt{2}\,(\varepsilon b)^{1/6}+\sqrt{3}\,\sqrt{\xi^2+(\varepsilon b)^{1/3}}\,)(\xi+\frac{(\varepsilon b)^{1/6}}{\sqrt{2}})}\right)\Bigg)$$
$$\left.\left.+\mathcal{O}\left(\frac{\tau^{-4/3}}{\left(\xi\pm\frac{1}{\sqrt{2}}(\varepsilon b)^{1/6}\right)^2}\right)+\mathcal{O}\left(\frac{\tau^{-4/3}}{\sqrt{\xi\pm\mathrm{i}(\varepsilon b)^{1/6}}}\right)\right)\right|_{\widetilde{\mu}^{\mathrm{TP}}}^{\widetilde{\mu}}. \tag{72}$$

*Proof.* Presenting $l(\widetilde{\mu})=l(\widetilde{\mu},\tau)=l(\widetilde{\mu},+\infty)(1+\Delta)^{1/2}$, where $\Delta:=\frac{l^2(\widetilde{\mu},\tau)-l^2(\widetilde{\mu},+\infty)}{l^2(\widetilde{\mu},+\infty)}$, approximating $(1+\Delta)^{1/2}$ as $1+\frac{1}{2}\Delta+\mathcal{O}(\Delta^2)$, integrating by parts, and using the following integrals [31], $\int\sqrt{x^2+a_o^2}\,\mathrm{d}x=\frac{x\sqrt{x^2+a_o^2}}{2}+\frac{a_o^2}{2}\ln(x+\sqrt{x^2+a_o^2})$, $\int\frac{\mathrm{d}x}{\sqrt{a_ox^2+b_ox+c_o}}=\frac{1}{\sqrt{a_o}}\ln(2\sqrt{a_o}\,\sqrt{a_ox^2+b_ox+c_o}$ $+2a_ox+b_o)$, and $\int\frac{\mathrm{d}x}{x\sqrt{a_ox^2+b_ox+c_o}}=-\frac{1}{\sqrt{c_o}}\ln\left(\frac{2\sqrt{c_o}\,\sqrt{a_ox^2+b_ox+c_o}+b_ox+2c_o}{x}\right)$, one arrives at the result stated in Equation (72). □

**Corollary 4.1.1.** *For* $\widetilde{\mu}^{\mathrm{TP}}=\frac{(\varepsilon b)^{1/6}}{\sqrt{2}}+\tau^{-1/3}\widetilde{\Lambda}$, *where* $\widetilde{\Lambda}=_{\tau\to+\infty}\mathcal{O}(\tau^{\epsilon_0})$, $0<\epsilon_0<\frac{1}{9}$,

$$\int_{\widetilde{\mu}^{\mathrm{TP}}}^{\widetilde{\mu}} l(\xi)\,\mathrm{d}\xi \underset{\tau\to+\infty}{=} \left(\widetilde{\mu}+\frac{(\varepsilon b)^{1/3}}{\widetilde{\mu}}\right)\sqrt{\widetilde{\mu}^2+(\varepsilon b)^{1/3}}+\tau^{-2/3}\left(\left(a-\frac{\mathrm{i}}{2}\right)\ln\left(2\sqrt{\widetilde{\mu}^2+(\varepsilon b)^{1/3}}+2\widetilde{\mu}\right)\right.$$
$$+\frac{1}{2\sqrt{3}}\left(\left(a-\frac{\mathrm{i}}{2}\right)+\frac{2h_0(\tau)}{(\varepsilon b)^{1/3}}\right)\ln\left(\frac{(-\widetilde{\mu}+\sqrt{2}\,(\varepsilon b)^{1/6}+\sqrt{3}\,\sqrt{\widetilde{\mu}^2+(\varepsilon b)^{1/3}}\,)(\widetilde{\mu}-\frac{(\varepsilon b)^{1/6}}{\sqrt{2}})}{(\widetilde{\mu}+\sqrt{2}\,(\varepsilon b)^{1/6}+\sqrt{3}\,\sqrt{\widetilde{\mu}^2+(\varepsilon b)^{1/3}}\,)(\widetilde{\mu}+\frac{(\varepsilon b)^{1/6}}{\sqrt{2}})}\right)\Bigg)$$
$$-\frac{3\sqrt{3}\,(\varepsilon b)^{1/3}}{2}-\left(a-\frac{\mathrm{i}}{2}\right)\tau^{-2/3}\ln\left(\sqrt{2}\,(\sqrt{3}+1)(\varepsilon b)^{1/6}\right)-2\sqrt{3}\,\tau^{-2/3}\widetilde{\Lambda}^2$$
$$+\frac{\tau^{-2/3}}{2\sqrt{3}}\left(\left(a-\frac{\mathrm{i}}{2}\right)+\frac{2h_0(\tau)}{(\varepsilon b)^{1/3}}\right)\left(\ln\left(\frac{3(\varepsilon b)^{1/6}}{\sqrt{2}}\right)-\ln\widetilde{\Lambda}+\frac{1}{3}\ln\tau\right)+o(\tau^{-2/3}). \tag{73}$$

*Proof.* Substituting into Equation (72) the expression for $\widetilde{\mu}^{\mathrm{TP}}$ given in the Corollary and expanding the result into a power series in $\widetilde{\Lambda}$, one gets for the error estimate $\mathcal{O}(\tau^{-1}\widetilde{\Lambda}^3)+\mathcal{O}(\tau^{-2/3}\widetilde{\Lambda}^{-2})+\mathcal{O}(\tau^{-1}\widetilde{\Lambda})$. □

**Corollary 4.1.2.** *For the choice of* $\widetilde{\mu}^{\mathrm{TP}}$ *given in Corollary* 4.1.1,

$$-\mathrm{i}\tau^{2/3}\int_{\widetilde{\mu}^{\mathrm{TP}}}^{\widetilde{\mu}} l(\xi)\,\mathrm{d}\xi \underset{\substack{\tau\to+\infty\\ \widetilde{\mu}\to\infty\\ \arg\widetilde{\mu}=0}}{=} -\mathrm{i}\left(\tau^{2/3}\widetilde{\mu}^2+\left(a-\frac{\mathrm{i}}{2}\right)\ln\widetilde{\mu}\right)+\frac{\mathrm{i}\tau^{2/3}(\varepsilon b)^{1/3}(3\sqrt{3}-2)}{2}+2\sqrt{3}\,\mathrm{i}\widetilde{\Lambda}^2$$
$$+\frac{\mathrm{i}}{2\sqrt{3}}\left(\left(a-\frac{\mathrm{i}}{2}\right)+\frac{2h_0(\tau)}{(\varepsilon b)^{1/3}}\right)\left(\ln\widetilde{\Lambda}-\frac{1}{3}\ln\tau\right)+C_{\infty}^{\mathrm{WKB}}+o(1), \tag{74}$$

*where*

$$C_{\infty}^{\mathrm{WKB}}:=\mathrm{i}\left(a-\frac{\mathrm{i}}{2}\right)\ln\left(\frac{(\sqrt{3}+1)(\varepsilon b)^{1/6}}{2\sqrt{2}}\right)-\frac{\mathrm{i}}{2\sqrt{3}}\left(\left(a-\frac{\mathrm{i}}{2}\right)+\frac{2h_0(\tau)}{(\varepsilon b)^{1/3}}\right)\ln\left(\frac{3\sqrt{2}\,(\varepsilon b)^{1/6}}{(\sqrt{3}+1)^2}\right), \tag{75}$$

*and*

$$-\mathrm{i}\tau^{2/3}\int_{\widetilde{\mu}^{\mathrm{TP}}}^{\widetilde{\mu}} l(\xi)\,\mathrm{d}\xi \underset{\substack{\tau\to+\infty\\ \widetilde{\mu}\to+0}}{=} \frac{\mathrm{i}\tau^{2/3}\sqrt{\varepsilon b}}{\widetilde{\mu}}-\frac{3\sqrt{3}\,\mathrm{i}(\varepsilon b)^{1/3}\tau^{2/3}}{2}-2\sqrt{3}\,\mathrm{i}\widetilde{\Lambda}^2-\frac{\mathrm{i}}{2\sqrt{3}}\left(\left(a-\frac{\mathrm{i}}{2}\right)\right.$$

$$+\frac{2h_0(\tau)}{(\varepsilon b)^{1/3}}\bigg)\bigg(\ln\widetilde{\Lambda}-\frac{1}{3}\ln\tau\bigg)+C_0^{\mathrm{WKB}}+o(1), \tag{76}$$

*where*

$$C_0^{\mathrm{WKB}}:=\mathrm{i}\bigg(a-\frac{\mathrm{i}}{2}\bigg)\ln\bigg(\frac{\sqrt{2}}{\sqrt{3}+1}\bigg)+\frac{\mathrm{i}}{2\sqrt{3}}\bigg(\bigg(a-\frac{\mathrm{i}}{2}\bigg)+\frac{2h_0(\tau)}{(\varepsilon b)^{1/3}}\bigg)\ln\bigg(\frac{3(\varepsilon b)^{1/6}}{\sqrt{2}}\bigg). \tag{77}$$

*Proof.* Equation (74) is the large-$\widetilde{\mu}$ asymptotics of Equation (73). Equation (76) is the small-$\widetilde{\mu}$ asymptotics of Equation (73), but the branch of $l(\xi)$ in Equation (76) is chosen differently than that in Equation (73). □

**Corollary 4.1.3.** *For* $\widetilde{\mu}^{\mathrm{TP}}=\mathrm{i}(\varepsilon b)^{1/6}+\tau^{-4/9}\widehat{\Lambda}$, *where* $\widehat{\Lambda}=_{\tau\to+\infty}\mathcal{O}(\tau^{\epsilon_0})$, $0<\epsilon_0<\frac{4}{9}$,

$$\begin{aligned}\int_{\widetilde{\mu}^{\mathrm{TP}}}^{\widetilde{\mu}} l(\xi)\,\mathrm{d}\xi \underset{\tau\to+\infty}{=} &\bigg(\widetilde{\mu}+\frac{(\varepsilon b)^{1/3}}{\widetilde{\mu}}\bigg)\sqrt{\widetilde{\mu}^2+(\varepsilon b)^{1/3}}+\tau^{-2/3}\bigg(\bigg(a-\frac{\mathrm{i}}{2}\bigg)\ln\Big(2\sqrt{\widetilde{\mu}^2+(\varepsilon b)^{1/3}}+2\widetilde{\mu}\Big)\\ &+\frac{1}{2\sqrt{3}}\bigg(\bigg(a-\frac{\mathrm{i}}{2}\bigg)+\frac{2h_0(\tau)}{(\varepsilon b)^{1/3}}\bigg)\ln\bigg(\frac{(-\widetilde{\mu}+\sqrt{2}\,(\varepsilon b)^{1/6}+\sqrt{3}\,\sqrt{\widetilde{\mu}^2+(\varepsilon b)^{1/3}}\,)(\widetilde{\mu}-\frac{(\varepsilon b)^{1/6}}{\sqrt{2}})}{(\widetilde{\mu}+\sqrt{2}\,(\varepsilon b)^{1/6}+\sqrt{3}\,\sqrt{\widetilde{\mu}^2+(\varepsilon b)^{1/3}}\,)(\widetilde{\mu}+\frac{(\varepsilon b)^{1/6}}{\sqrt{2}})}\bigg)\bigg)\\ &+2\sqrt{2}\,\mathrm{i}^{-3/2}(\varepsilon b)^{1/12}\tau^{-2/3}\widehat{\Lambda}^{3/2}-\tau^{-2/3}\big(a-\tfrac{\mathrm{i}}{2}\big)\ln\Big(2\mathrm{i}(\varepsilon b)^{1/6}\Big)+o(\tau^{-2/3}).\end{aligned} \tag{78}$$

*Proof.* Analogous to the proof of Corollary 4.1.1; but in this case, for the error estimate, one obtains $\mathcal{O}(\tau^{-10/9}\widehat{\Lambda}^{-1/2})+\mathcal{O}(\tau^{-8/9}\widehat{\Lambda}^{1/2})$. □

**Corollary 4.1.4.** *For the choice of* $\widetilde{\mu}^{\mathrm{TP}}$ *given in Corollary* 4.1.3*,*

$$\begin{aligned}-\mathrm{i}\tau^{2/3}\int_{\widetilde{\mu}^{\mathrm{TP}}}^{\widetilde{\mu}} l(\xi)\,\mathrm{d}\xi \underset{\substack{\tau\to+\infty\\ \widetilde{\mu}\to\infty\\ \arg\widetilde{\mu}=\frac{\pi}{2}}}{=} &-\mathrm{i}\Big(\tau^{2/3}\widetilde{\mu}^2+(a-\tfrac{\mathrm{i}}{2})\ln\widetilde{\mu}\Big)-\mathrm{i}\tau^{2/3}(\varepsilon b)^{1/3}-2\sqrt{2}\,\mathrm{i}^{-1/2}(\varepsilon b)^{1/12}\widehat{\Lambda}^{3/2}\\ &+\mathrm{i}\big(a-\tfrac{\mathrm{i}}{2}\big)\Big(\ln\Big(\tfrac{(\varepsilon b)^{1/6}}{2}\Big)+\tfrac{\pi\mathrm{i}}{2}\Big)+\Big(\tfrac{\mathrm{i}}{2\sqrt{3}}\big(a-\tfrac{\mathrm{i}}{2}\big)+\tfrac{2h_0(\tau)}{(\varepsilon b)^{1/3}}\Big)\ln\Big(\tfrac{\sqrt{3}+1}{\sqrt{3}-1}\Big)+o(1).\end{aligned} \tag{79}$$

*Proof.* Equation (79) is the large-$\widetilde{\mu}$ asymptotics of Equation (78). □

**Proposition 4.1.3.**

$$T(\widetilde{\mu}) \underset{\substack{\widetilde{\mu}\to\infty\\ \arg\widetilde{\mu}=0}}{=} (b(\tau))^{-\frac{1}{2}\sigma_3}\big(\mathrm{I}+\mathcal{O}\big(\widetilde{\mu}^{-1}\big)\big)(b(\tau))^{\frac{1}{2}\sigma_3}, \tag{80}$$

$$T(\widetilde{\mu}) \underset{\widetilde{\mu}\to+0}{=} (b(\tau))^{-\frac{1}{2}\sigma_3}\bigg(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & (\varepsilon b)^{1/2}\\ -(\varepsilon b)^{-1/2} & 1\end{pmatrix}+\mathcal{O}(\widetilde{\mu})\bigg)(b(\tau))^{\frac{1}{2}\sigma_3}. \tag{81}$$

*Proof.* Asymptotics (80) and (81) are obtained from Equation (70) upon choosing the same branches of $l(\widetilde{\mu})$ as in Equations (74) and (76), respectively. □

**Remark 4.1.2.** The elements of the matrices $\mathcal{O}(\widetilde{\mu}^{-1})$ and $\mathcal{O}(\widetilde{\mu})$ in Equations (80) and (81), respectively, as functions of $\tau$, behave like $\mathcal{O}(1)$ as $\tau\to+\infty$. ■

**Proposition 4.1.4.** *For* $0<\delta_{\diamondsuit}<\epsilon_0<\frac{1}{9}$, *set* $\widetilde{\mu}=\frac{(\varepsilon b)^{1/6}}{\sqrt{2}}+\tau^{-1/3}\widetilde{\Lambda}$, *where* $\widetilde{\Lambda}=_{\tau\to+\infty}\mathcal{O}(\tau^{\epsilon_0})$, *and* $\arg\widetilde{\Lambda}=\frac{1}{2}(1-\widetilde{\varepsilon}_1)\pi$, $\widetilde{\varepsilon}_1=\pm1$. *Then,*

$$T(\widetilde{\mu}) \underset{\tau\to+\infty}{=} \frac{(b(\tau))^{-\frac{1}{2}\sigma_3}}{\sqrt{2\sqrt{3}\,(\sqrt{3}+\widetilde{\varepsilon}_1)}}\bigg(\begin{pmatrix}\sqrt{3}+\widetilde{\varepsilon}_1 & -\widetilde{\varepsilon}_1\sqrt{2\varepsilon b}\\ \frac{\widetilde{\varepsilon}_1\sqrt{2}}{\sqrt{\varepsilon b}} & \sqrt{3}+\widetilde{\varepsilon}_1\end{pmatrix}+\mathcal{O}\Big(\tau^{-(\epsilon_0-\delta_{\diamondsuit})}\Big)\bigg)(b(\tau))^{\frac{1}{2}\sigma_3}.$$

*Proof.* The result follows from the definition of $T(\widetilde{\mu})$ given in Equation (70) and conditions (61) and (63). $\square$

**Proposition 4.1.5.** *For the conditions stated in Proposition* 4.1.4*,*

$$\int_{\widetilde{\mu}^{\mathrm{TP}}}^{\widetilde{\mu}} \operatorname{diag}\big((T(\xi))^{-1}\partial_{\xi}T(\xi)\big)\,\mathrm{d}\xi \underset{\tau\to+\infty}{=} o(1)\sigma_3. \tag{82}$$

*Proof.* Using the definition of $T(\widetilde{\mu})$ given in Equation (70), one shows that the integrand is proportional to the $\xi$-independent factor $\mathcal{D}(\tau):=4\sqrt{-a(\tau)b(\tau)}+\frac{r(\tau)(\varepsilon b)^{4/3}}{2\sqrt{-a(\tau)b(\tau)}}$: this factor, as $\tau\to+\infty$, is $o(1)$ due to conditions (61) and (63). $\square$

## 4.2 Asymptotics Near the Turning Points

For the calculation of the monodromy data, one needs a more accurate approximation for the solution of Equation (57) in the neighbourhoods of the turning points than that given by the WKB formula (cf. Equation (68)). For the pair of coalescing turning points (resp., single turning points), this approximation is known to be given in terms of parabolic cylinder (resp., Airy) functions (see, for example, [28, 29]): more accurate statements are given below.

**Proposition 4.2.1.** *Set* $\widetilde{\mu}=\frac{(\varepsilon b)^{1/6}}{\sqrt{2}}+\tau^{-1/3}\widetilde{\Lambda}$*, where* $\arg\widetilde{\Lambda}\in\{0,\pi\}$*, and* $|\widetilde{\Lambda}|<_{\tau\to+\infty}\mathcal{O}(\tau^{\epsilon_0})$*,* $0<\epsilon_0<\frac{1}{9}$*. Then, under conditions* (60)–(64)*, for any fundamental solution of Equation* (57) *the following asymptotic representation is valid,*

$$\widetilde{\Psi}(\widetilde{\mu}) \underset{\tau\to+\infty}{=} \mathcal{N}(\tau)\big(\mathrm{I}+\mathcal{O}\big(\tau^{-\epsilon}\big)\big)\widetilde{\Psi}_0(\widetilde{\Lambda}), \tag{83}$$

*where*

$$\mathcal{N}(\tau):=\frac{\mathrm{i}\sqrt{b(\tau)}}{(6\varepsilon b)^{1/4}}\begin{pmatrix} -\frac{(\sqrt{3}-1)\sqrt{\varepsilon b}}{\sqrt{2}\,b(\tau)} & \frac{\sqrt{2}\sqrt{\varepsilon b}}{(\sqrt{3}-1)b(\tau)} \\ 1 & 1 \end{pmatrix}, \tag{84}$$

$\epsilon$ *is some positive number, and* $\widetilde{\Psi}_0(\widetilde{\Lambda})$ *is a fundamental solution of*

$$\frac{\partial\widetilde{\Psi}_0(\widetilde{\Lambda})}{\partial\widetilde{\Lambda}}=\left(4\sqrt{3}\,\mathrm{i}\widetilde{\Lambda}\sigma_3+\begin{pmatrix}0 & \widetilde{p}\\ \widetilde{q} & 0\end{pmatrix}\right)\widetilde{\Psi}_0(\widetilde{\Lambda}), \tag{85}$$

*with*

$$\begin{aligned}\widetilde{p}&:=\frac{\mathrm{i}(\varepsilon b)^{1/6}(\sqrt{3}+1)(4u_0(\tau)+(\sqrt{3}+1)r_0(\tau))}{2\sqrt{2}},\\ \widetilde{q}&:=-\frac{\mathrm{i}(\varepsilon b)^{1/6}(\sqrt{3}-1)(4u_0(\tau)-(\sqrt{3}-1)r_0(\tau))}{2\sqrt{2}}.\end{aligned} \tag{86}$$

*Proof.* Making the change of variables $(\widetilde{\Psi},\,\widetilde{\mu})\to(\widetilde{\Psi}^{\natural},\,\widetilde{\Lambda})$ in Equation (57), one shows that

$$\frac{\partial\widetilde{\Psi}^{\natural}(\widetilde{\Lambda})}{\partial\widetilde{\Lambda}} \underset{\tau\to+\infty}{=} \Big(P_1^{\natural}\widetilde{\Lambda}+P_0^{\natural}+(b(\tau))^{-\frac{1}{2}\sigma_3}\mathcal{O}\Big(\tau^{-1/3}\widetilde{\Lambda}^2\Big)(b(\tau))^{\frac{1}{2}\sigma_3}\Big)\widetilde{\Psi}^{\natural}(\widetilde{\Lambda}),$$

where

$$P_1^{\natural}:=\begin{pmatrix}\mathrm{i}(r(\tau)-2) & -\frac{4\sqrt{2}\,\mathrm{i}(\varepsilon b)^{1/2}}{b(\tau)}\\ -\frac{4\sqrt{2}\,\mathrm{i}b(\tau)}{(\varepsilon b)^{1/2}} & -\mathrm{i}(r(\tau)-2)\end{pmatrix},$$

$$P_0^{\natural}:=\begin{pmatrix}-\frac{\mathrm{i}(\varepsilon b)^{1/6}(r(\tau)+2)\tau^{1/3}}{\sqrt{2}} & -\frac{4\mathrm{i}\sqrt{-a(\tau)b(\tau)}\,\tau^{1/3}}{b(\tau)}+\frac{2\mathrm{i}(\varepsilon b)^{2/3}\tau^{1/3}}{b(\tau)}\\ \frac{2\mathrm{i}b(\tau)\tau^{1/3}}{(\varepsilon b)^{1/3}}+\frac{\mathrm{i}b(\tau)r(\tau)(\varepsilon b)^{1/3}\tau^{1/3}}{2\sqrt{-a(\tau)b(\tau)}} & \frac{\mathrm{i}(\varepsilon b)^{1/6}(r(\tau)+2)\tau^{1/3}}{\sqrt{2}}\end{pmatrix}.$$

Now, from conditions (61) and (63), one finds that

$$P_1^{\natural}=-4\mathrm{i}\begin{pmatrix} 1 & \frac{\sqrt{2}\,(\varepsilon b)^{1/2}}{b(\tau)} \\ \frac{\sqrt{2}\,b(\tau)}{(\varepsilon b)^{1/2}} & -1 \end{pmatrix}+\mathrm{i}r_0(\tau)\tau^{-1/3}\sigma_3,$$

$$P_0^{\natural}=\begin{pmatrix} -\frac{\mathrm{i}(\varepsilon b)^{1/6}r_0(\tau)}{\sqrt{2}} & -\frac{2\mathrm{i}(\varepsilon b)^{2/3}u_0(\tau)}{b(\tau)} \\ \frac{2\mathrm{i}b(\tau)}{(\varepsilon b)^{1/3}}\Big(u_0(\tau)+\frac{r_0(\tau)}{2}\Big) & \frac{\mathrm{i}(\varepsilon b)^{1/6}r_0(\tau)}{\sqrt{2}} \end{pmatrix}+b(\tau)\mathcal{E}\sigma_-,$$

where

$$\mathcal{E}:=\mathcal{O}\Big(r_0(\tau)u_0^2(\tau)\tau^{-\frac{2}{3}}\Big)+\mathcal{O}\Big(r_0(\tau)u_0(\tau)\tau^{-\frac{1}{3}}\Big)+\mathcal{O}\Big(u_0^2(\tau)\tau^{-\frac{1}{3}}\Big)+\mathcal{O}\Big(u_0(\tau)\tau^{-\frac{2}{3}}\Big)+\mathcal{O}\Big(\tau^{-\frac{1}{3}}\Big).$$

Making the gauge transformation

$$\widetilde{\Psi}^{\natural}(\widetilde{\Lambda})=\mathcal{N}(\tau)\widehat{\Psi}(\widetilde{\Lambda}),$$

where $\mathcal{N}(\tau)$ is given by Equation (84), one arrives at

$$\frac{\partial\widehat{\Psi}(\widetilde{\Lambda})}{\partial\widetilde{\Lambda}}\underset{\tau\to+\infty}{=}\left(4\sqrt{3}\,\mathrm{i}\widetilde{\Lambda}\sigma_3+\begin{pmatrix} 0 & \widetilde{p} \\ \widetilde{q} & 0\end{pmatrix}+\mathrm{i}r_0(\tau)\tau^{-1/3}\widetilde{\Lambda}\sigma_3+\mathcal{E}\sigma_-+\mathcal{O}\Big(\tau^{-1/3}\widetilde{\Lambda}^2\Big)\right)\widehat{\Psi}(\widetilde{\Lambda}).$$

From Equation (67), it follows that $\widetilde{p}\widetilde{q}=_{\tau\to+\infty}\mathcal{O}(1)$. Since one of either $\widetilde{p}$ or $\widetilde{q}$ exhibits the power-like growth $\mathcal{O}(\tau^{\delta_\diamondsuit})$, the other decays like $\mathcal{O}(\tau^{-\delta_\diamondsuit})$. Using the fact that (cf. Equation (64)) $\delta_\diamondsuit\in(0,\frac{1}{9})$, $\widetilde{\Lambda}$ is real, and $|\widetilde{\Lambda}|\leqslant_{\tau\to+\infty}\mathcal{O}(\tau^{\epsilon_0})$, one proceeds analogously as in [15, 22] to prove that

$$\widehat{\Psi}(\widetilde{\Lambda})\underset{\tau\to+\infty}{=}\big(\mathrm{I}+\mathcal{O}\big(\tau^{-\epsilon}\big)\big)\widetilde{\Psi}_0(\widetilde{\Lambda}),$$

where $\widetilde{\Psi}_0(\widetilde{\Lambda})$ is a fundamental solution of Equation (85). □

**Proposition 4.2.2.** *A fundamental solution of Equation* (85) *is*

$$\widetilde{\Psi}_0(\widetilde{\Lambda})=\begin{pmatrix} D_{-1-\nu}\Big(2\sqrt{2\sqrt{3}}\,\mathrm{i}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\widetilde{\Lambda}\Big) & D_{\nu}\Big(2\sqrt{2\sqrt{3}}\,\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\widetilde{\Lambda}\Big) \\ \eth_{\mathcal{D}}D_{-1-\nu}\Big(2\sqrt{2\sqrt{3}}\,\mathrm{i}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\widetilde{\Lambda}\Big) & \eth_{\mathcal{D}}D_{\nu}\Big(2\sqrt{2\sqrt{3}}\,\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\widetilde{\Lambda}\Big)\end{pmatrix},$$

*where* $\eth_{\mathcal{D}}:=(\widetilde{p})^{-1}(\partial_{\widetilde{\Lambda}}-4\sqrt{3}\,\mathrm{i}\widetilde{\Lambda})$*, and* $D_\star(\cdot)$ *is the parabolic cylinder function* [31].

*Proof.* Changing the independent variable according to the rule $\widetilde{\Lambda}=\alpha x$, where $\alpha:=\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}\Big(2\sqrt{2\sqrt{3}}\Big)^{-1}$, one shows that $\widetilde{\Psi}_0(\widetilde{\Lambda}):=\mathcal{D}(x)$ satisfies $\partial_x\mathcal{D}(x)=\Big(\frac{x}{2}\sigma_3+\big(\begin{smallmatrix} 0 & p^* \\ q^* & 0\end{smallmatrix}\big)\Big)\mathcal{D}(x)$, with $(p^*,q^*):=(\alpha\widetilde{p},\alpha\widetilde{q})$, whose fundamental solution is [15, 22] $\mathcal{D}(x)=\Big(\begin{smallmatrix} D_{-1-\nu}(\mathrm{i}x) & D_\nu(x) \\ \dot{D}_{-1-\nu}(\mathrm{i}x) & \dot{D}_\nu(x)\end{smallmatrix}\Big)$, where $\dot{D}_\star(z):=(p^*)^{-1}(\partial_z D_\star(z)-\frac{z}{2}D_\star(z))$, $D_\star(\cdot)$ is the parabolic cylinder function, and

$$\nu+1:=-p^*q^*\underset{\tau\to+\infty}{=}-\frac{\mathrm{i}}{2\sqrt{3}}\left(\Big(a-\frac{\mathrm{i}}{2}\Big)+\frac{2h_0(\tau)}{(\varepsilon b)^{1/3}}\right)\Big(1+\mathcal{O}\Big(\tau^{-\frac{1}{3}+3\delta_\diamondsuit}\Big)\Big)\underset{\tau\to+\infty}{=}\mathcal{O}(1). \qquad \square \quad (87)$$

**Remark 4.2.1.** In the proof of Proposition 4.2.2, the matrix-valued function $\mathcal{D}(x)$ is introduced. In Subsection 4.3, the following asymptotics of $\mathcal{D}(x)$ as $|x|\to\infty$ are required (see, for example, [23]):

$$\mathcal{D}(x)\underset{\substack{x\to\infty \\ \arg x=\frac{\pi\mathrm{i}}{4}+k\frac{\pi\mathrm{i}}{2}}}{=}\big(\mathrm{I}+\mathcal{O}\big(x^{-1}\big)\big)\exp\left(\Big(\frac{x^2}{4}-(\nu+1)\ln x\Big)\sigma_3\right)\mathcal{R}_k,\quad k=-1,0,1,2,$$

where

$$\mathcal{R}_{-1}:=\begin{pmatrix}\mathrm{e}^{-\frac{\pi\mathrm{i}}{2}(\nu+1)} & 0\\ 0 & -(p^*)^{-1}\end{pmatrix},\qquad \mathcal{R}_0:=\begin{pmatrix}\mathrm{e}^{-\frac{\pi\mathrm{i}}{2}(\nu+1)} & 0\\ -\frac{\mathrm{i}}{p^*}\frac{\sqrt{2\pi}}{\Gamma(\nu+1)}\mathrm{e}^{-\frac{\pi\mathrm{i}}{2}(\nu+1)} & -(p^*)^{-1}\end{pmatrix},$$

$$\mathcal{R}_1:=\begin{pmatrix}\mathrm{e}^{\frac{3\pi\mathrm{i}}{2}(\nu+1)} & \frac{\sqrt{2\pi}}{\Gamma(-\nu)}\mathrm{e}^{\pi\mathrm{i}(\nu+1)}\\ -\frac{\mathrm{i}}{p^*}\frac{\sqrt{2\pi}}{\Gamma(\nu+1)}\mathrm{e}^{-\frac{\pi\mathrm{i}}{2}(\nu+1)} & -(p^*)^{-1}\end{pmatrix},\qquad \mathcal{R}_2:=\begin{pmatrix}\mathrm{e}^{\frac{3\pi\mathrm{i}}{2}(\nu+1)} & \frac{\sqrt{2\pi}}{\Gamma(-\nu)}\mathrm{e}^{\pi\mathrm{i}(\nu+1)}\\ 0 & -\frac{1}{p^*}\mathrm{e}^{-2\pi\mathrm{i}(\nu+1)}\end{pmatrix},$$

and $\Gamma(\cdot)$ is the gamma function [31]. ■

### 4.3 Asymptotic Matching as $\tau\to+\infty$

Let $\widetilde{\Psi}(\widetilde{\mu})$ be a fundamental solution of Equation (57) corresponding to (see Equation (83)) the function $\widetilde{\Psi}_0(\widetilde{\Lambda})$ given in Proposition 4.2.2. Define $\widetilde{Y}_0^{\infty}(\widetilde{\mu}):=\tau^{-(1/12)\sigma_3}Y_0^{\infty}(\widetilde{\mu}\tau^{-1/6})$, where $Y_0^{\infty}(\cdot)$ is the canonical solution of Equation (55) (cf. Proposition 2.2), and

$$L_{\infty}:=\left(\widetilde{\Psi}(\widetilde{\mu})\right)^{-1}\widetilde{Y}_0^{\infty}(\widetilde{\mu}).$$

**Lemma 4.3.1.** *Let conditions* (60)–(64) *be valid, and*

$$b(\tau)\tau^{-\frac{1}{3}\mathrm{Im}(a)}\underset{\tau\to+\infty}{=}\mathcal{O}(1). \tag{88}$$

*Then there exists $\epsilon_{\nu}^{\infty}>0$ and $\delta_{\infty}>0$ such that for*

$$|\mathrm{Re}(\nu+1)|\underset{\tau\to+\infty}{<}\epsilon_{\nu}^{\infty}, \tag{89}$$

$$\begin{aligned}L_{\infty}\underset{\tau\to+\infty}{=}&-\mathrm{i}(\mathcal{R}_0)^{-1}\exp\left(\left(\frac{(3\sqrt{3}-2)\mathrm{i}(\varepsilon b)^{1/3}\tau^{2/3}}{2}+\left(-\frac{\mathrm{i}a}{6}+\frac{(\nu+1)}{3}\right)\ln\tau+\frac{\mathrm{i}\pi(\nu+1)}{4}\right.\right.\\ &\left.\left.+\ \widetilde{C}_{\infty}^{\mathrm{WKB}}\right)\sigma_3\right)\left(\frac{(\varepsilon b)^{1/4}\sqrt{\sqrt{3}+1}}{2^{1/4}\sqrt{b(\tau)}}\right)^{\sigma_3}\sigma_1\left(\mathrm{I}+\mathcal{O}\left(\tau^{-\delta_{\infty}}\right)\right),\end{aligned} \tag{90}$$

*where $\mathcal{R}_0$ is defined in Remark* 4.2.1*, and*

$$\begin{aligned}\widetilde{C}_{\infty}^{\mathrm{WKB}}:=&\left(-\frac{3\mathrm{i}}{2}\left(a-\frac{\mathrm{i}}{2}\right)+2(\nu+1)\right)\ln 2+\frac{5}{4}(\nu+1)\ln 3+\frac{1}{6}\left(\mathrm{i}\left(a-\frac{\mathrm{i}}{2}\right)+(\nu+1)\right)\ln(\varepsilon b)\\ &+\left(\mathrm{i}\left(a-\frac{\mathrm{i}}{2}\right)-2(\nu+1)\right)\ln\left(\sqrt{3}+1\right).\end{aligned} \tag{91}$$

*Proof.* Denote by $\widetilde{\Psi}_{\mathrm{WKB}}(\widetilde{\mu})$ the solution of Equation (57) which has WKB asymptotics (68) in the canonical domain containing the Stokes curve approaching the positive $\widetilde{\mu}$-real axis from above as $\widetilde{\mu}\to\infty$. Rewriting $L_{\infty}$ as $L_{\infty}=\left((\widetilde{\Psi}(\widetilde{\mu}))^{-1}\widetilde{\Psi}_{\mathrm{WKB}}(\widetilde{\mu})\right)\left((\widetilde{\Psi}_{\mathrm{WKB}}(\widetilde{\mu}))^{-1}\widetilde{Y}_0^{\infty}(\widetilde{\mu})\right)$, and noting that the members of the pairs in parentheses are solutions of the same equation (Equation (57)), they differ by right-hand, $\widetilde{\mu}$-independent matrices: the latter matrices are calculated by taking different $\widetilde{\mu}$ limits ($\widetilde{\mu}\to+\infty$ and $\widetilde{\mu}\to(\varepsilon b)^{1/6}/\sqrt{2}$). More precisely, modulo factors that are $(\mathrm{I}+\mathcal{O}(\tau^{-\widetilde{\delta}}))$, for some $\widetilde{\delta}>0$,

$$L_{\infty}\underset{\tau\to+\infty}{=}(\underbrace{(\mathcal{N}(\tau)\widetilde{\Psi}_0(\widetilde{\Lambda}))^{-1}T(\widetilde{\mu})}_{\substack{\widetilde{\Lambda}\sim\tau^{\epsilon_o},\quad \arg\widetilde{\Lambda}=0\\ \widetilde{\mu}=\frac{(\varepsilon b)^{1/6}}{\sqrt{2}}+\tau^{-1/3}\widetilde{\Lambda}}})(\underbrace{(\widetilde{\Psi}_{\mathrm{WKB}}(\widetilde{\mu}))^{-1}\widetilde{Y}_0^{\infty}(\widetilde{\mu})}_{\widetilde{\mu}\to\infty,\quad \arg\widetilde{\mu}=0}),$$

where $\mathcal{N}(\tau)$ and $\widetilde{\Psi}_0(\widetilde{\Lambda})$, respectively, are given in Propositions 4.2.1 and 4.2.2 (see, also, Remark 4.2.1), $T(\widetilde{\mu})$ is given in Proposition 4.1.4, $\widetilde{\Psi}_{\mathrm{WKB}}(\widetilde{\mu})$ is defined by Equation (68) (with $T(\widetilde{\mu})$ given in Equation (80) and the phase-part of the WKB formula estimated in Equations (74), (75), and (82)), and $\widetilde{Y}_0^{\infty}(\cdot)$ is defined in the beginning of this subsection. The additional conditions (88) and (89) are necessary in order to guarantee the power-like decay, $\mathcal{O}(\tau^{-\delta_\infty})$, $\delta_\infty>0$, of the off-diagonal entries of $L_\infty$ given in Equation (90). □

Define $\widetilde{X}_0^0(\widetilde{\mu}):=\tau^{-(1/12)\sigma_3}X_0^0(\widetilde{\mu}\tau^{-1/6})$, where $X_0^0(\cdot)$ is the canonical solution of Equation (55) (cf. Proposition 2.2), and

$$L_0:=\left(\widetilde{\Psi}(\widetilde{\mu})\right)^{-1}\widetilde{X}_0^0(\widetilde{\mu}).$$

**Lemma 4.3.2.** *Under conditions* (60)–(64) *and* (89), *there exists* $\epsilon_\nu^0>0$ *and* $\delta_0>0$ *such that for*

$$|\mathrm{Re}(\nu+1)| \underset{\tau\to+\infty}{<} \epsilon_\nu^0, \tag{92}$$

$$\begin{aligned} L_0 \underset{\tau\to+\infty}{=} & -\mathrm{i}(\mathcal{R}_2)^{-1}\exp\left(\left(\frac{3\sqrt{3}\,\mathrm{i}(\varepsilon b)^{1/3}\tau^{2/3}}{2}+\frac{(\nu+1)}{3}\ln\tau+\frac{\mathrm{i}\pi(\nu+1)}{4}+\widetilde{C}_0^{\mathrm{WKB}}\right)\sigma_3\right)\\ &\times\left(\frac{\sqrt{\sqrt{3}+1}}{2^{1/4}}\right)^{\sigma_3}\sigma_2\left(\mathrm{I}+\mathcal{O}\left(\tau^{-\delta_0}\right)\right), \end{aligned}\tag{93}$$

*where* $\mathcal{R}_2$ *is defined in Remark* 4.2.1, *and*

$$\widetilde{C}_0^{\mathrm{WKB}}:=\left(-\frac{\mathrm{i}}{2}\left(a-\frac{\mathrm{i}}{2}\right)+(\nu+1)\right)\ln 2+\frac{5}{4}(\nu+1)\ln 3+\frac{1}{6}(\nu+1)\ln(\varepsilon b)+\mathrm{i}\left(a-\frac{\mathrm{i}}{2}\right)\ln\left(\sqrt{3}+1\right). \tag{94}$$

*Proof.* Denote by $\widetilde{\Psi}_{\mathrm{WKB}}(\widetilde{\mu})$ the solution of Equation (57) which has WKB asymptotics (68) in the canonical domain containing the Stokes curve approaching the positive $\widetilde{\mu}$-real axis from above as $\widetilde{\mu}\to 0$. Rewriting $L_0$ as $L_0=\left((\widetilde{\Psi}(\widetilde{\mu}))^{-1}\widetilde{\Psi}_{\mathrm{WKB}}(\widetilde{\mu})\right)\left((\widetilde{\Psi}_{\mathrm{WKB}}(\widetilde{\mu}))^{-1}\widetilde{X}_0^0(\widetilde{\mu})\right)$, and arguing analogously as in the proof of Lemma 4.3.1, one can estimate $L_0$ by taking different $\widetilde{\mu}$ limits ($\widetilde{\mu}\to+0$ and $\widetilde{\mu}\to(\varepsilon b)^{1/6}/\sqrt{2}$). More precisely, modulo factors that are $(\mathrm{I}+\mathcal{O}(\tau^{-\widehat{\delta}}))$, for some $\widehat{\delta}>0$,

$$L_0 \underset{\tau\to+\infty}{=} (\underbrace{(\mathcal{N}(\tau)\widetilde{\Psi}_0(\widetilde{\Lambda}))^{-1}T(\widetilde{\mu})}_{\substack{\widetilde{\Lambda}\sim\tau^{\epsilon_o},\quad \arg\widetilde{\Lambda}=\pi\\ \widetilde{\mu}=\frac{(\varepsilon b)^{1/6}}{\sqrt{2}}+\tau^{-1/3}\widetilde{\Lambda}}})(\underbrace{(\widetilde{\Psi}_{\mathrm{WKB}}(\widetilde{\mu}))^{-1}\widetilde{X}_0^0(\widetilde{\mu})}_{\widetilde{\mu}\to 0,\quad \arg\widetilde{\mu}=0}),$$

where $\mathcal{N}(\tau)$ and $\widetilde{\Psi}_0(\widetilde{\Lambda})$, respectively, are given in Propositions 4.2.1 and 4.2.2 (see, also, Remark 4.2.1), $T(\widetilde{\mu})$ is given in Proposition 4.1.4, $\widetilde{\Psi}_{\mathrm{WKB}}(\widetilde{\mu})$ is defined by Equation (68) (with $T(\widetilde{\mu})$ given in Equation (81) and the phase-part of the WKB formula estimated in Equations (76), (77), and (82)), and $\widetilde{X}_0^0(\cdot)$ is defined in the paragraph preceding Lemma 4.3.2. □

**Theorem 4.3.1.** *Under conditions* (60)–(64), (88), (89), *and* (92), *the connection matrix* $G$ *(cf. Equation* (28)*) has the following asymptotics,*

$$\begin{aligned} G \underset{\tau\to+\infty}{=} & \begin{pmatrix} -\frac{\mathrm{i}\mathrm{e}^{-2\pi\mathrm{i}(\nu+1)}\sqrt{b(\tau)}\,\mathrm{e}^{(z_2-z_1)}}{(\varepsilon b)^{1/4}} & \frac{\sqrt{2\pi}\,(\varepsilon b)^{1/4}\sqrt{2+\sqrt{3}}\,\mathrm{e}^{-2\pi\mathrm{i}(\nu+1)}\mathrm{e}^{(z_2+z_1)}}{p^*\sqrt{b(\tau)}\,\Gamma(\nu+1)}\\ -\frac{\mathrm{i}p^*\sqrt{2\pi}\,\mathrm{e}^{\pi\mathrm{i}(\nu+1)}\sqrt{b(\tau)}\,\mathrm{e}^{-(z_2+z_1)}}{(\varepsilon b)^{1/4}\sqrt{2+\sqrt{3}}\,\Gamma(-\nu)} & \frac{\mathrm{i}(\varepsilon b)^{1/4}\mathrm{e}^{-(z_2-z_1)}}{\sqrt{b(\tau)}} \end{pmatrix}\\ &\times\left(\mathrm{I}+\mathcal{O}\left(\tau^{-\delta}\right)\right), \end{aligned}\tag{95}$$

*where*

$$z_2-z_1:=\mathrm{i}(\varepsilon b)^{1/3}\tau^{2/3}+\frac{\mathrm{i}a}{6}\ln\tau+\mathrm{i}\left(a-\frac{\mathrm{i}}{2}\right)\ln 2-\frac{\mathrm{i}}{6}\left(a-\frac{\mathrm{i}}{2}\right)\ln(\varepsilon b)+(\nu+1)\ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right), \tag{96}$$

$$\begin{aligned}z_2+z_1:=&\left(3\sqrt{3}-1\right)\mathrm{i}(\varepsilon b)^{1/3}\tau^{2/3}+\left(\frac{2}{3}(\nu+1)-\frac{\mathrm{i}a}{6}\right)\ln\tau+\frac{\mathrm{i}\pi}{2}(\nu+1)+\frac{5}{2}(\nu+1)\ln 3\\&+\left(-\mathrm{i}\left(a-\frac{\mathrm{i}}{2}\right)+2(\nu+1)\right)\ln 2+\frac{1}{6}\left(\mathrm{i}\left(a-\frac{\mathrm{i}}{2}\right)+2(\nu+1)\right)\ln(\varepsilon b)+\left(\mathrm{i}\left(a-\frac{\mathrm{i}}{2}\right)\right.\\&-\left.(\nu+1)\right)\ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right),\end{aligned} \tag{97}$$

*and* $\delta>0$; *in particular,* $G\underset{\tau\to+\infty}{=}\mathcal{O}(1)$.

*Proof.* From the definition of $G$ (Equation (28)), $L_\infty$, and $L_0$, one arrives at $G=(L_0)^{-1}L_\infty$: now, via straightforward calculations, the result stated in the Theorem is a consequence of Lemmae 4.3.1 and 4.3.2. □

**Remark 4.3.1.** It follows from Equation (28) that $\det(G)=1$: one shows, via the well-known identity $\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}$, that asymptotics (95) is consistent with this fact. ■

**Corollary 4.3.1.** *Let* $g_{ij}$, $i,j=1,2$, *be the matrix elements of* $G$. *There exists* $\epsilon_\nu>0$ *such that for*

$$|\arg(g_{11}g_{22})|<\epsilon_\nu, \qquad g_{11}g_{12}g_{21}g_{22}\neq 0, \tag{98}$$

*the functions* $b(\tau)$, $r_0(\tau)$, $u_0(\tau)$, *and* $h_0(\tau)$ *have the following asymptotic representation,*

$$\begin{aligned}b(\tau)\underset{\tau\to+\infty}{=}&-g_{11}^2\sqrt{\varepsilon b}\,\mathrm{e}^{4\pi\mathrm{i}(\nu+1)}\exp\left(-2\left(\mathrm{i}(\varepsilon b)^{1/3}\tau^{2/3}+\frac{\mathrm{i}a}{6}\ln\tau+\mathrm{i}\left(a-\frac{\mathrm{i}}{2}\right)\ln 2\right.\right.\\&-\left.\left.\frac{\mathrm{i}}{6}\left(a-\frac{\mathrm{i}}{2}\right)\ln(\varepsilon b)+(\nu+1)\ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)+o\left(\tau^{-\delta}\right)\right)\right),\end{aligned} \tag{99}$$

$$r_0(\tau)\underset{\tau\to+\infty}{=}\frac{4\sqrt{2}\,\sqrt{\nu+1}\,\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}}{3^{1/4}(\varepsilon b)^{1/6}}\sinh\left(\ln\widehat{x}-2\pi\mathrm{i}(\nu+1)+\frac{1}{2}\ln\left(2+\sqrt{3}\right)+\mathrm{i}\widehat{y}+z_n+o\left(\tau^{-\delta}\right)\right), \tag{100}$$

$$u_0(\tau)\underset{\tau\to+\infty}{=}\frac{2\sqrt{\nu+1}\,\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}}{3^{1/4}(\varepsilon b)^{1/6}}\cosh\left(\ln\widehat{x}-2\pi\mathrm{i}(\nu+1)+\mathrm{i}\widehat{y}+z_n+o\left(\tau^{-\delta}\right)\right), \tag{101}$$

$$h_0(\tau)\underset{\tau\to+\infty}{=}-\frac{(\varepsilon b)^{1/3}}{2}\left(\frac{\sqrt{3}}{\pi}\ln(g_{11}g_{22})+\left(a-\frac{\mathrm{i}}{2}\right)+o\left(\tau^{-\delta}\right)\right), \tag{102}$$

*where*

$$\widehat{x}:=\left|\frac{g_{22}g_{21}\Gamma(-(\nu+1))}{g_{11}g_{12}\Gamma(\nu+1)}\right|^{1/2}, \qquad \widehat{y}:=\frac{1}{2}\arg\left(\frac{g_{22}g_{21}\Gamma(-(\nu+1))}{g_{11}g_{12}\Gamma(\nu+1)}\right),$$

$$\begin{aligned}z_n:=&3\sqrt{3}\,\mathrm{i}(\varepsilon b)^{1/3}\tau^{2/3}+\frac{2}{3}(\nu+1)\ln\tau+\frac{\pi\mathrm{i}}{4}+2(\nu+1)\ln 2+\frac{5}{2}(\nu+1)\ln 3\\&+\frac{1}{3}(\nu+1)\ln(\varepsilon b)+\mathrm{i}a\ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right),\end{aligned}$$

*and* $\delta>0$.

*Proof.* Multiplying the diagonal elements of $G$ (Equation (95)), one arrives at

$$(\nu+1)=\frac{\mathrm{i}}{2\pi}\ln(g_{11}g_{22})\left(1+o\left(\tau^{-\delta}\right)\right): \tag{103}$$

taking conditions (89) and (92) into account, and setting $\epsilon_\nu := 2\pi\min\{\epsilon_\nu^\infty, \epsilon_\nu^0\}$, one obtains the first of conditions (98). Equation (99) is obtained from the (1 1)-element of $G$. From the definition of $p^*$ and $q^*$ given in the proof of Proposition 4.2.2 and Equations (86), one obtains

$$r_0(\tau)=\frac{4\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}}{3^{1/4}(\varepsilon b)^{1/6}}\left(\frac{p^*}{\sqrt{3}+1}+\frac{q^*}{\sqrt{3}-1}\right), \qquad u_0(\tau)=\frac{\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}}{3^{1/4}(\varepsilon b)^{1/6}}\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}p^*-\frac{\sqrt{3}+1}{\sqrt{3}-1}q^*\right).$$

Using the second of conditions (98), $p^*q^* = -(\nu+1)$ (cf. Equation (87)), and the identity $\Gamma(-\nu)=-(\nu+1)\Gamma(-(\nu+1))$, one deduces, from the second, respectively, first, column of $G$, the following representation for $p^*$, respectively, $q^*$:

$$p^*=-\mathrm{i}\sqrt{2\pi(2+\sqrt{3})}\,\frac{g_{22}\mathrm{e}^{-2\pi\mathrm{i}(\nu+1)}\mathrm{e}^{2z_2}}{g_{12}\Gamma(\nu+1)}, \qquad q^*=\frac{\sqrt{2\pi}\,g_{11}\mathrm{e}^{3\pi\mathrm{i}(\nu+1)}\mathrm{e}^{-2z_2}}{\sqrt{2+\sqrt{3}}\,g_{21}\Gamma(-(\nu+1))}.$$

Substituting the latter formulae into the relations for $r_0(\tau)$ and $u_0(\tau)$ given above, one obtains Equations (100) and (101). Equation (102) is a direct consequence of Equations (87) and (103).
□

**Proposition 4.3.1.** *Let $G$ be the connection matrix of Equation* (55) *with $\tau$-independent elements satisfying conditions* (98)*. Then the corresponding isomonodromy deformations have the following asymptotic representation,*

$$\sqrt{-a(\tau)b(\tau)} \underset{\tau\to+\infty}{=} \frac{(\varepsilon b)^{2/3}}{2}+\frac{\sqrt{\varepsilon b(\nu+1)}\,\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}}{3^{1/4}\tau^{1/3}}\cosh\Big(\ln\widehat{x}-2\pi\mathrm{i}(\nu+1)+\mathrm{i}\widehat{y}+z_n+o\big(\tau^{-\delta}\big)\Big), \quad (104)$$

$$\begin{aligned} a(\tau)d(\tau) \underset{\tau\to+\infty}{=} & -\frac{\mathrm{i}\varepsilon b}{4}+\frac{\mathrm{i}(\varepsilon b)^{5/6}\sqrt{\nu+1}\,\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}}{2\cdot 3^{1/4}\tau^{1/3}}\Big(\sqrt{2}\sinh\Big(\ln\widehat{x}-2\pi\mathrm{i}(\nu+1)+\frac{1}{2}\ln\big(2+\sqrt{3}\big) \\ & +\mathrm{i}\widehat{y}+z_n+o\big(\tau^{-\delta}\big)\Big)-\cosh\Big(\ln\widehat{x}-2\pi\mathrm{i}(\nu+1)+\mathrm{i}\widehat{y}+z_n+o\big(\tau^{-\delta}\big)\Big)\Big), \end{aligned} \quad (105)$$

$$\begin{aligned} b(\tau)c(\tau) \underset{\tau\to+\infty}{=} & -\frac{\mathrm{i}\varepsilon b}{4}-\frac{\mathrm{i}(\varepsilon b)^{5/6}\sqrt{\nu+1}\,\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}}{2\cdot 3^{1/4}\tau^{1/3}}\Big(\sqrt{2}\sinh\Big(\ln\widehat{x}-2\pi\mathrm{i}(\nu+1)+\frac{1}{2}\ln\big(2+\sqrt{3}\big) \\ & +\mathrm{i}\widehat{y}+z_n+o\big(\tau^{-\delta}\big)\Big)-\cosh\Big(\ln\widehat{x}-2\pi\mathrm{i}(\nu+1)+\mathrm{i}\widehat{y}+z_n+o\big(\tau^{-\delta}\big)\Big)\Big), \end{aligned} \quad (106)$$

$$c(\tau)d(\tau) \underset{\tau\to+\infty}{=} \frac{(\varepsilon b)^{2/3}}{4}-\frac{\sqrt{\varepsilon b(\nu+1)}\,\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}}{3^{1/4}\tau^{1/3}}\cosh\Big(\ln\widehat{x}-2\pi\mathrm{i}(\nu+1)+\mathrm{i}\widehat{y}+z_n+o\big(\tau^{-\delta}\big)\Big), \quad (107)$$

*and $b(\tau)$ is given in Equation* (99)*.*

*Proof.* If the elements of $G$ are $\tau$-independent, then any functions whose asymptotics (as $\tau\to+\infty$) are given by Equations (99)–(102) satisfy conditions (60)–(64), (88), (89), and (92); therefore, one can now use the justification scheme suggested in [24] (see, also, [26]). Equations (104)–(107) are obtained from Equations (62), (63), (65), and (66) by the direct substitution of Equations (100)–(102). □

According to Proposition 1.2, $u(\tau)$, the solution of the degenerate third Painlevé equation (1), can be written in terms of the functions $a(\tau)$ and $b(\tau)$ as

$$u(\tau)=\varepsilon\tau^{1/3}\sqrt{-a(\tau)b(\tau)}, \quad \varepsilon=\pm 1$$

(cf. Equations (56)); therefore, the result (as $\tau\to+\infty$) stated in Theorem 3.1 follows from Equation (104).

Comparing Equation (37) with Equation (62), and taking into account Equation (56), one shows that

$$\mathcal{H}(\tau) \underset{\tau\to+\infty}{=} 3(\varepsilon b)^{2/3}\tau^{1/3}-4h_0(\tau)\tau^{-1/3}+\frac{(a-\mathrm{i}/2)^2}{2\tau}:$$

now, the asymptotics of $\mathcal{H}(\tau)$ given in Theorem 3.1, Equation (40) follows from Equations (87) and (103).

# 5 Calculation of the Monodromy Data as $\tau \to +0$

In this section, the first equation of System (4) is studied asymptotically as $\tau \to +0$ under specific conditions on the elements of $\mathcal{U}(\lambda,\tau)$. Fundamental solutions of this equation in the neighbourhoods of the essential singularities (0 and $\infty$) are approximated in terms of Hankel and Whittaker functions: this allows one to calculate, asymptotically, the corresponding monodromy data.

Denoting by $\mathcal{U}_0$ and $\mathcal{V}_0$ the following matrices,

$$\mathcal{U}_0=\tau\left(-\mathrm{i}\sigma_3-\frac{a\mathrm{i}}{2\tau\lambda}\sigma_3-\frac{1}{\lambda}\begin{pmatrix}0 & C\\ D & 0\end{pmatrix}\right), \qquad \mathcal{V}_0=\frac{\mathrm{i}\tau}{2\lambda^2}\begin{pmatrix}\sqrt{-AB} & A\\ B & -\sqrt{-AB}\end{pmatrix},$$

the first equation of System (4) can be rewritten as

$$\partial_\lambda\Phi(\lambda)=(\mathcal{U}_0+\mathcal{V}_0)\Phi(\lambda). \tag{108}$$

**Proposition 5.1.** *Consider*

$$\partial_\lambda\mathbf{W}(\lambda)=\mathcal{U}_0\mathbf{W}(\lambda). \tag{109}$$

*A fundamental solution of Equation* (109) *is given by*

$$\mathbf{W}(\lambda)=\frac{\mathrm{e}^{-\frac{\pi a}{4}}}{\sqrt{2\mathrm{i}\lambda\tau}}\begin{pmatrix}W_{\varkappa_1,\widehat{\rho}}(2\mathrm{i}\lambda\tau) & \mathrm{i}\widehat{\gamma}W_{-\varkappa_1,\widehat{\rho}}(-2\mathrm{i}\lambda\tau)\\ \widehat{\delta}\,W_{\varkappa_1-1,\widehat{\rho}}(2\mathrm{i}\lambda\tau) & \mathrm{i}W_{-(\varkappa_1-1),\widehat{\rho}}(-2\mathrm{i}\lambda\tau)\end{pmatrix}\mathrm{e}^{\mathrm{i}a\ln(\sqrt{2\tau})\sigma_3}, \tag{110}$$

*where*

$$\varkappa_1:=\frac{1}{2}(1-a\mathrm{i}), \qquad \widehat{\rho}^2:=\widehat{\gamma}\,\widehat{\delta}-\frac{a^2}{4}, \qquad \widehat{\gamma}:=\tau C, \qquad \widehat{\delta}:=\tau D, \tag{111}$$

*and* $W_{\star,\widehat{\rho}}(\cdot)$ *is the Whittaker function* [31]*; moreover,* $\det(\mathbf{W}(\lambda))=1$*, and*

$$\mathbf{W}(\lambda)\underset{\substack{\lambda\to\infty\\ \arg\lambda=0}}{=}\left(\mathrm{I}+\frac{1}{2\mathrm{i}\tau\lambda}\begin{pmatrix}\widehat{\gamma}\,\widehat{\delta} & -\widehat{\gamma}\\ \widehat{\delta} & -\widehat{\gamma}\,\widehat{\delta}\end{pmatrix}+\mathcal{O}\left(\frac{1}{\lambda^2}\right)\right)\mathrm{e}^{-\mathrm{i}(\tau\lambda+\frac{a}{2}\ln\lambda)\sigma_3}. \tag{112}$$

*Proof.* Rewriting Equation (109) as a system for its components, one deduces from it, for $(\mathbf{W}(\lambda))_{11}$ and $(\mathbf{W}(\lambda))_{22}$, the Whittaker ODE: then, $(\mathbf{W}(\lambda))_{12}$ and $(\mathbf{W}(\lambda))_{21}$ are obtained from the latter system by applying certain identities for the Whittaker function. □

**Proposition 5.2.** *Let* $\epsilon_i>0$*,* $i=1,2$*, and the parameters of Equation* (108) *satisfy the following restrictions:*

$$\begin{gathered}|\mathrm{Im}(a)|<1, \qquad \widehat{\rho}\underset{\tau\to+0}{=}\mathcal{O}(1), \qquad A\tau^{1+\mathrm{i}a}\underset{\tau\to+0}{\sim}\tau^{\epsilon_1},\\ B\tau^{1-\mathrm{i}a}\underset{\tau\to+0}{\sim}\tau^{\epsilon_1}, \qquad |C|\underset{\tau\to+0}{>}|A|\tau^{-\epsilon_1}, \qquad |D|\underset{\tau\to+0}{>}|B|\tau^{-\epsilon_1}.\end{gathered} \tag{113}$$

*Then there exists a fundamental solution of Equation* (108) *which has the asymptotic representation*

$$\Phi(\lambda)\underset{\substack{\tau\to+0\\ \lambda>\epsilon_2,\ \arg\lambda=0}}{=}\mathbf{W}(\lambda)\left(\mathrm{I}+o\left(\left(\frac{\tau}{\lambda}\right)^{\delta_1}\right)\right), \tag{114}$$

*with* $\delta_1>0$.

*Proof.* Follows from a successive approximations argument applied to Equation (108). □

For the purpose of approximating the solution of the first equation of System (4) as $\lambda\to0$, it is convenient to rewrite it as

$$\partial_\lambda\Phi(\lambda)=(\widetilde{\mathcal{U}}_0+\widetilde{\mathcal{V}}_0)\Phi(\lambda), \tag{115}$$

where

$$\widetilde{\mathcal{U}}_0=\tau\left(-\frac{a\mathrm{i}}{2\tau\lambda}\sigma_3-\frac{1}{\lambda}\begin{pmatrix}0 & C\\ D & 0\end{pmatrix}+\frac{\mathrm{i}}{2\lambda^2}\begin{pmatrix}\sqrt{-AB} & A\\ B & -\sqrt{-AB}\end{pmatrix}\right), \qquad \widetilde{\mathcal{V}}_0=-\mathrm{i}\tau\sigma_3:$$

this equation is compared to the following model system,

$$\partial_\lambda\mathfrak{B}(\lambda)=\widetilde{\mathcal{U}}_0\mathfrak{B}(\lambda). \tag{116}$$

**Proposition 5.3.** *The fundamental solution of Equation* (116) *can be written as follows:*

$$\mathfrak{B}(\lambda)=\begin{pmatrix}\mathfrak{B}_{11}(\lambda) & \mathfrak{B}_{12}(\lambda)\\ \mathfrak{B}_{21}(\lambda) & \mathfrak{B}_{22}(\lambda)\end{pmatrix}, \tag{117}$$

*where*

$$\begin{aligned}
\mathfrak{B}_{11}(\lambda)&=-\frac{\sqrt{-AB}\,\sqrt{\pi}\,\mathrm{e}^{-\mathrm{i}(\frac{\pi\nu_o}{2}+\frac{\pi}{4})}}{2\sqrt{\varepsilon b}\,\sqrt{B}}\left(\mathfrak{r}^{\uparrow}(\tau)H^{(2)}_{\nu_o}\left(\sqrt{\frac{\tau\varepsilon b}{\lambda}}\right)+\sqrt{\frac{\tau\varepsilon b}{\lambda}}H^{(2)}_{\nu_o-1}\left(\sqrt{\frac{\tau\varepsilon b}{\lambda}}\right)\right),\\
\mathfrak{B}_{12}(\lambda)&=-\frac{\sqrt{-AB}\,\sqrt{\pi}\,\mathrm{e}^{\mathrm{i}(\frac{\pi\nu_o}{2}+\frac{\pi}{4})}}{2\sqrt{\varepsilon b}\,\sqrt{B}}\left(\mathfrak{r}^{\uparrow}(\tau)H^{(1)}_{\nu_o}\left(\sqrt{\frac{\tau\varepsilon b}{\lambda}}\right)+\sqrt{\frac{\tau\varepsilon b}{\lambda}}H^{(1)}_{\nu_o-1}\left(\sqrt{\frac{\tau\varepsilon b}{\lambda}}\right)\right),\\
\mathfrak{B}_{21}(\lambda)&=-\frac{\sqrt{B}\,\sqrt{\pi}\,\mathrm{e}^{-\mathrm{i}(\frac{\pi\nu_o}{2}+\frac{\pi}{4})}}{2\sqrt{\varepsilon b}}\left(\mathfrak{r}^{\downarrow}(\tau)H^{(2)}_{\nu_o}\left(\sqrt{\frac{\tau\varepsilon b}{\lambda}}\right)+\sqrt{\frac{\tau\varepsilon b}{\lambda}}H^{(2)}_{\nu_o-1}\left(\sqrt{\frac{\tau\varepsilon b}{\lambda}}\right)\right),\\
\mathfrak{B}_{22}(\lambda)&=-\frac{\sqrt{B}\,\sqrt{\pi}\,\mathrm{e}^{\mathrm{i}(\frac{\pi\nu_o}{2}+\frac{\pi}{4})}}{2\sqrt{\varepsilon b}}\left(\mathfrak{r}^{\downarrow}(\tau)H^{(1)}_{\nu_o}\left(\sqrt{\frac{\tau\varepsilon b}{\lambda}}\right)+\sqrt{\frac{\tau\varepsilon b}{\lambda}}H^{(1)}_{\nu_o-1}\left(\sqrt{\frac{\tau\varepsilon b}{\lambda}}\right)\right),
\end{aligned}$$

*with*

$$\mathfrak{r}^{\uparrow}(\tau):=-\nu_o+a\mathrm{i}+\frac{2B\widehat{\gamma}}{\sqrt{-AB}}, \qquad \mathfrak{r}^{\downarrow}(\tau):=-\nu_o-a\mathrm{i}+\frac{2\widehat{\delta}\sqrt{-AB}}{B}, \qquad \nu_o^2:=4\widehat{\rho}^2, \tag{118}$$

*and* $H^{(j)}_{\star}(\cdot)$, $j=1,2$, *the Hankel functions of the first* $(j=1)$ *and second* $(j=2)$ *kind* [31].

*Furthermore,* $\det(\mathfrak{B}(\lambda))=1$, *and* $\mathfrak{B}(\lambda)$ *has the asymptotic expansion*

$$\begin{aligned}
\mathfrak{B}(\lambda)\underset{\substack{\lambda\to+0\\ \arg\lambda=0}}{=}&\left(\frac{\mathrm{i}\tau^{1/4}\lambda^{-1/4}}{\sqrt{2}\,(\varepsilon b)^{1/4}}\begin{pmatrix}\frac{\sqrt{-AB}}{\sqrt{B}} & -\frac{\sqrt{-AB}}{\sqrt{B}}\\ \sqrt{B} & -\sqrt{B}\end{pmatrix}+\frac{\lambda^{1/4}\tau^{-1/4}}{\sqrt{2}\,(\varepsilon b)^{3/4}}\right.\\
&\times\begin{pmatrix}-\frac{\sqrt{-AB}}{\sqrt{B}}\left(\mathfrak{r}^{\uparrow}(\tau)-\frac{(4(\nu_o-1)^2-1)}{8}\right) & -\frac{\sqrt{-AB}}{\sqrt{B}}\left(\mathfrak{r}^{\uparrow}(\tau)-\frac{(4(\nu_o-1)^2-1)}{8}\right)\\ -\sqrt{B}\left(\mathfrak{r}^{\uparrow}(\tau)-\frac{(4(\nu_o-1)^2-1)}{8}\right)-\frac{\mathrm{i}\varepsilon b\sqrt{B}}{\sqrt{-AB}} & -\sqrt{B}\left(\mathfrak{r}^{\uparrow}(\tau)-\frac{(4(\nu_o-1)^2-1)}{8}\right)-\frac{\mathrm{i}\varepsilon b\sqrt{B}}{\sqrt{-AB}}\end{pmatrix}\\
&\left.+\,\mathcal{O}\left(\frac{\lambda^{3/4}}{\tau^{3/4}}\right)\right)\exp\left(-\mathrm{i}\sqrt{\frac{\tau\varepsilon b}{\lambda}}\,\sigma_3\right).
\end{aligned} \tag{119}$$

*Proof.* Defining the functions $\widetilde{\Phi}^{(0)}_{\uparrow\downarrow}(x)$ by $\mathfrak{B}(x):=\widehat{\phi}^{0}_{\uparrow\downarrow}\widetilde{\Phi}^{(0)}_{\uparrow\downarrow}(x)$, where $x:=\mathrm{i}\tau/\lambda$, $\widehat{\phi}^0_{\uparrow}=\begin{pmatrix}1 & 0\\ \frac{B}{\sqrt{-AB}} & 1\end{pmatrix}$, and $\widehat{\phi}^0_{\downarrow}=\begin{pmatrix}1 & \frac{\sqrt{-AB}}{B}\\ 0 & 1\end{pmatrix}$, one arrives at

$$\partial_x\widetilde{\Phi}^{(0)}_{\uparrow}(x)=\left(\frac{a\mathrm{i}}{2x}\begin{pmatrix}1 & 0\\ -\frac{2B}{\sqrt{-AB}} & -1\end{pmatrix}+\frac{\tau}{x}\begin{pmatrix}\frac{BC}{\sqrt{-AB}} & C\\ \frac{(AD+BC)}{A} & -\frac{BC}{\sqrt{-AB}}\end{pmatrix}-\frac{1}{2}\begin{pmatrix}0 & A\\ 0 & 0\end{pmatrix}\right)\widetilde{\Phi}^{(0)}_{\uparrow}(x), \tag{120}$$

$$\partial_x\widetilde{\Phi}^{(0)}_{\downarrow}(x)=\left(\frac{a\mathrm{i}}{2x}\begin{pmatrix}1 & \frac{2\sqrt{-AB}}{B}\\ 0 & -1\end{pmatrix}+\frac{\tau}{x}\begin{pmatrix}-\frac{D\sqrt{-AB}}{B} & \frac{(AD+BC)}{B}\\ D & \frac{D\sqrt{-AB}}{B}\end{pmatrix}-\frac{1}{2}\begin{pmatrix}0 & 0\\ B & 0\end{pmatrix}\right)\widetilde{\Phi}^{(0)}_{\downarrow}(x). \tag{121}$$

Consider, for example, Equation (120). Rewriting it in component form, one obtains (modulo the change $x \to (-\mathrm{i}\varepsilon b x)^{1/2}$) for them the Bessel equation. Choosing

$$\left(\widetilde{\Phi}_{\uparrow}^{(0)}(\lambda)\right)_{21} = c_{21} H_{\nu_o}^{(2)}\left(\sqrt{(\tau\varepsilon b)/\lambda}\right) \qquad \text{and} \qquad \left(\widetilde{\Phi}_{\uparrow}^{(0)}(\lambda)\right)_{22} = c_{22} H_{\nu_o}^{(1)}\left(\sqrt{(\tau\varepsilon b)/\lambda}\right),$$

where $c_{21} = -\frac{\mathrm{i}\sqrt{\pi\varepsilon b}\sqrt{B}\,\mathrm{e}^{-\mathrm{i}(\frac{\pi\nu_o}{2}+\frac{\pi}{4})}}{2\sqrt{-AB}}$, $c_{22} = -\frac{\mathrm{i}\sqrt{\pi\varepsilon b}\sqrt{B}\,\mathrm{e}^{\mathrm{i}(\frac{\pi\nu_o}{2}+\frac{\pi}{4})}}{2\sqrt{-AB}}$, and $H_{\star}^{(j)}(\cdot)$ are the Hankel functions of the first ($j=1$) and second ($j=2$) kind, one obtains

$$\left(\widetilde{\Phi}_{\uparrow}^{(0)}(\lambda)\right)_{11} = \frac{\mathrm{i}c_{21}A}{\varepsilon b}\left(\mathfrak{r}^{\uparrow}(\tau) H_{\nu_o}^{(2)}\left(\sqrt{(\tau\varepsilon b)/\lambda}\right) + \sqrt{(\tau\varepsilon b)/\lambda}\, H_{\nu_o-1}^{(2)}\left(\sqrt{(\tau\varepsilon b)/\lambda}\right)\right)$$

and

$$\left(\widetilde{\Phi}_{\uparrow}^{(0)}(\lambda)\right)_{12} = \frac{\mathrm{i}c_{22}A}{\varepsilon b}\left(\mathfrak{r}^{\uparrow}(\tau) H_{\nu_o}^{(1)}\left(\sqrt{(\tau\varepsilon b)/\lambda}\right) + \sqrt{(\tau\varepsilon b)/\lambda}\, H_{\nu_o-1}^{(1)}\left(\sqrt{(\tau\varepsilon b)/\lambda}\right)\right)$$

from the corresponding component equations of Equation (120) and the identity $z\partial_z H_{\star}^{(j)}(z) = -\star\, H_{\star}^{(j)}(z) + z H_{\star-1}^{(j)}(z)$, $j=1,2$. The components $(\widetilde{\Phi}_{\uparrow}^{(0)}(\lambda))_{21}$ and $(\widetilde{\Phi}_{\uparrow}^{(0)}(\lambda))_{22}$ are chosen in such a way that the leading term of asymptotics of $\mathfrak{B}(\lambda)$ as $\lambda \to +0$ (Equation (119)) matches with the canonical asymptotics given in Equation (16): to verify this, one uses the following asymptotic expansions for the Hankel functions [32],

$$H_{\nu_o}^{(1)}(z) \underset{\substack{z\to\infty \\ |\arg z|<\pi}}{=} \sqrt{\frac{2}{\pi z}}\,\mathrm{e}^{\mathrm{i}(z-\frac{\pi\nu_o}{2}-\frac{\pi}{4})}\left(1 + \frac{\mathrm{i}(4\nu_o^2-1)}{8z} + \mathcal{O}\left(\frac{1}{z^2}\right)\right),$$

$$H_{\nu_o}^{(2)}(z) \underset{\substack{z\to\infty \\ |\arg z|<\pi}}{=} \sqrt{\frac{2}{\pi z}}\,\mathrm{e}^{-\mathrm{i}(z-\frac{\pi\nu_o}{2}-\frac{\pi}{4})}\left(1 - \frac{\mathrm{i}(4\nu_o^2-1)}{8z} + \mathcal{O}\left(\frac{1}{z^2}\right)\right).$$

Noting that $\det(\mathfrak{B}(\lambda)) = \det(\widetilde{\Phi}_{\uparrow}^{(0)}(\lambda))$ and the trace of the coefficient matrix of Equation (120) is zero, one shows that $\det(\mathfrak{B}(\lambda)) = \text{const.}$: one proves that const. $=1$ by calculating $\det(\mathfrak{B}(\lambda))$ as $\lambda \to +0$ by means of asymptotics (119). □

**Proposition 5.4.** *Let $0<\epsilon_i<1$, $i=3,4$. If the parameters of Equation* (115) *satisfy the restrictions*

$$\nu_o \underset{\tau\to+0}{=} \mathcal{O}(1), \qquad |AD+BC| \underset{\tau\to+0}{>} \left|\sqrt{-AB}\right|\tau^{-\epsilon_3}, \tag{122}$$

*and either*

$$\tau^{-\epsilon_3} \underset{\tau\to+0}{<} \frac{|BC|}{\left|\sqrt{-AB}\right|} \underset{\tau\to+0}{\leqslant} \mathcal{O}\left(\frac{1}{\tau}\right), \tag{123}$$

*or*

$$\tau^{-\epsilon_3} \underset{\tau\to+0}{<} \frac{D\sqrt{-AB}}{B} \underset{\tau\to+0}{\leqslant} \mathcal{O}\left(\frac{1}{\tau}\right), \tag{124}$$

*then there exists a fundamental solution of Equation* (115) *which has the asymptotic representation*

$$\Phi(\lambda) \underset{\substack{\tau\to+0 \\ 0<\lambda<\epsilon_4,\ \arg\lambda=0}}{=} \mathfrak{B}(\lambda)\left(\mathrm{I} + o\left((\tau\lambda)^{\delta}\right)\right), \tag{125}$$

*with some $\delta>0$.*

*Proof.* Consider, for example, the restrictions (122) and (123) imposed on the coefficients of Equation (115). Consider the transformation $\Phi(\lambda) = \widehat{\phi}_{\uparrow}^{0}\widetilde{\Phi}_{\uparrow}(\lambda)$, where $\widehat{\phi}_{\uparrow}^{0}$ is defined in the proof of Proposition 5.3. Note that, under the assumed conditions, $(\widehat{\phi}_{\uparrow}^{0})^{-1}\widetilde{\mathcal{V}}_0\widehat{\phi}_{\uparrow}^{0} \underset{\substack{\tau\to+0 \\ 0<\lambda<\epsilon_4}}{=} \mathcal{O}\left(\tau^{-\epsilon}(\widehat{\phi}_{\uparrow}^{0})^{-1}\widetilde{\mathfrak{U}}_0\widehat{\phi}_{\uparrow}^{0}\right)$, with some $\epsilon_4, \epsilon > 0$: now, using a successive approximations argument,

one shows that $\widetilde{\Phi}_{\uparrow}(\lambda) \underset{\substack{\tau\to+0\\0<\lambda<\epsilon_4,\ \arg\lambda=0}}{=} \widetilde{\Phi}_{\uparrow}^{(0)}(\lambda)\big(1+o((\tau\lambda)^{\delta})\big)$. For the restrictions (122) and (124), one proceeds as above, but, instead of for $\widetilde{\Phi}_{\uparrow}(\lambda)$, for the function $\widetilde{\Phi}_{\downarrow}(\lambda)=(\widehat{\phi}_{\downarrow}^{0})^{-1}\Phi(\lambda)$, where $\widehat{\phi}_{\downarrow}^{0}$ is defined in the proof of Proposition 5.3. □

**Lemma 5.1.** *Let* $0<\epsilon<1$,

$$\widehat{\rho} \underset{\tau\to+0}{\neq} 0, \qquad |\mathrm{Re}(\widehat{\rho})| \underset{\tau\to+0}{<} \frac{1}{2},$$
$$\left|\sqrt{-AB}\right|+|A/\widehat{\gamma}| \underset{\tau\to+0}{>} (|B\,\widehat{\gamma}|+\mathcal{O}(1))\tau^{2-4|\mathrm{Re}(\widehat{\rho})|},$$
$$\left|\sqrt{-AB}\right|+\left|A\,\widehat{\delta}\right| \underset{\tau\to+0}{>} (|B\,\widehat{\gamma}|+\mathcal{O}(1))\tau^{2-4|\mathrm{Re}(\widehat{\rho})|},$$
$$\left|\sqrt{-AB}\right|+\left|B/\widehat{\delta}\right| \underset{\tau\to+0}{>} \left(\left|A\widehat{\delta}\right|+\mathcal{O}(1)\right)\tau^{2-4|\mathrm{Re}(\widehat{\rho})|},$$
$$\left|\sqrt{-AB}\right|+|B\,\widehat{\gamma}| \underset{\tau\to+0}{>} \left(\left|A\widehat{\delta}\right|+\mathcal{O}(1)\right)\tau^{2-4|\mathrm{Re}(\widehat{\rho})|},$$
$$\left|\widehat{\gamma}\sqrt{B}\,\tau^{-\frac{a\mathrm{i}}{2}}\right| \underset{\tau\to+0}{\leqslant} \mathcal{O}\left(\tau^{-2+2|\mathrm{Re}(\widehat{\rho})|+\epsilon}\right), \qquad \left|\sqrt{B}\,\tau^{\frac{a\mathrm{i}}{2}}\right| \underset{\tau\to+0}{\leqslant} \mathcal{O}\left(\tau^{-2+2|\mathrm{Re}(\widehat{\rho})|+\epsilon}\right).$$

*Then, under conditions* (113), (122) *and* (123), *or* (113), (122) *and* (124), *the connection matrix, G (Equation* (28)), *has the following asymptotics* ($g_{ij}:=(G)_{ij}$, $i,j=1,2$),

$$\begin{aligned} g_{11} \underset{\tau\to+0}{=} &\left(\sum_{l=\pm1}\frac{2^{l\widehat{\rho}+1}\mathrm{e}^{\frac{\mathrm{i}\pi l\widehat{\rho}}{2}}(\Gamma(2l\widehat{\rho}))^2\tau^{-2l\widehat{\rho}}(2\tau)^{\frac{a\mathrm{i}}{2}}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}l\widehat{\rho}}{\sqrt{\pi\varepsilon b}\,(\varepsilon b)^{l\widehat{\rho}}\big(l\widehat{\rho}+\frac{a\mathrm{i}}{2}\big)\mathrm{e}^{\frac{\pi a}{4}}\Gamma\big(l\widehat{\rho}+\frac{a\mathrm{i}}{2}\big)}\left(\sqrt{B}\left(l\widehat{\rho}+\frac{a\mathrm{i}}{2}\right)-\frac{\widehat{\delta}\sqrt{-AB}}{\sqrt{B}}\right)\right.\\ &\left.+\ o\left(\tau^{\delta_1}\right)\right)\left(1+o\left(\tau^{\delta_2}\right)\right), \end{aligned} \tag{126}$$

$$\begin{aligned} g_{12} \underset{\tau\to+0}{=} &\left(\sum_{l=\pm1}\frac{2^{l\widehat{\rho}+1}\mathrm{e}^{\frac{3\pi\mathrm{i}l\widehat{\rho}}{2}}(\Gamma(2l\widehat{\rho}))^2\tau^{-2l\widehat{\rho}}(2\tau)^{-\frac{a\mathrm{i}}{2}}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}l\widehat{\rho}}{\sqrt{\pi\varepsilon b}\,(\varepsilon b)^{l\widehat{\rho}}\big(l\widehat{\rho}-\frac{a\mathrm{i}}{2}\big)\mathrm{e}^{\frac{\pi a}{4}}\Gamma\big(l\widehat{\rho}-\frac{a\mathrm{i}}{2}\big)}\left(\sqrt{B}\,\widehat{\gamma}-\frac{\sqrt{-AB}}{\sqrt{B}}\left(l\widehat{\rho}-\frac{a\mathrm{i}}{2}\right)\right)\right.\\ &\left.+\ o\left(\tau^{\delta_1}\right)\right)\left(1+o\left(\tau^{\delta_2}\right)\right), \end{aligned} \tag{127}$$

$$\begin{aligned} g_{21} \underset{\tau\to+0}{=} &\left(-\sum_{l=\pm1}\frac{2^{l\widehat{\rho}+1}\mathrm{e}^{-\frac{3\pi\mathrm{i}l\widehat{\rho}}{2}}(\Gamma(2l\widehat{\rho}))^2\tau^{-2l\widehat{\rho}}(2\tau)^{\frac{a\mathrm{i}}{2}}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}l\widehat{\rho}}{\sqrt{\pi\varepsilon b}\,(\varepsilon b)^{l\widehat{\rho}}\big(l\widehat{\rho}+\frac{a\mathrm{i}}{2}\big)\mathrm{e}^{\frac{\pi a}{4}}\Gamma\big(l\widehat{\rho}+\frac{a\mathrm{i}}{2}\big)}\left(\sqrt{B}\left(l\widehat{\rho}+\frac{a\mathrm{i}}{2}\right)-\frac{\widehat{\delta}\sqrt{-AB}}{\sqrt{B}}\right)\right.\\ &\left.+\ o\left(\tau^{\delta_1}\right)\right)\left(1+o\left(\tau^{\delta_2}\right)\right), \end{aligned} \tag{128}$$

$$\begin{aligned} g_{22} \underset{\tau\to+0}{=} &\left(-\sum_{l=\pm1}\frac{2^{l\widehat{\rho}+1}\mathrm{e}^{-\frac{\mathrm{i}\pi l\widehat{\rho}}{2}}(\Gamma(2l\widehat{\rho}))^2\tau^{-2l\widehat{\rho}}(2\tau)^{-\frac{a\mathrm{i}}{2}}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}l\widehat{\rho}}{\sqrt{\pi\varepsilon b}\,(\varepsilon b)^{l\widehat{\rho}}\big(l\widehat{\rho}-\frac{a\mathrm{i}}{2}\big)\mathrm{e}^{\frac{\pi a}{4}}\Gamma\big(l\widehat{\rho}-\frac{a\mathrm{i}}{2}\big)}\left(\sqrt{B}\,\widehat{\gamma}-\frac{\sqrt{-AB}}{\sqrt{B}}\left(l\widehat{\rho}-\frac{a\mathrm{i}}{2}\right)\right)\right.\\ &\left.+\ o\left(\tau^{\delta_1}\right)\right)\left(1+o\left(\tau^{\delta_2}\right)\right), \end{aligned} \tag{129}$$

*with* $\delta_j>0$, $j=1,2$.

*Proof.* Consider, say, the derivation of the results for $G$ corresponding to conditions (113), (122) and (123): the derivation for the case corresponding to conditions (113), (122) and (124) is analogous. For this purpose, the results stated in Propositions 5.1 and 5.3 are applicable; therefore,

$$G \underset{\tau\to+0}{=} \left(1+o\left((\tau\lambda)^{\delta_2}\right)\right)(\mathfrak{B}(\lambda))^{-1}\,\mathbf{W}(\lambda)\left(1+o\left((\tau/\lambda)^{\delta_1}\right)\right), \tag{130}$$

with $\delta_j>0$, $j=1,2$. This formula allows one to calculate $G$ to the desired order by taking (real) $\lambda$ in the finite domain $\epsilon_2<\lambda<\epsilon_1$ and applying small-argument asymptotic expansions of the Whittaker and Hankel functions. Towards this end, one notes that [31] $H_{\star}^{(1)}(z)=$

$(2/\pi z)^{1/2}\,\exp\big(-\mathrm{i}\big(\frac{\pi\star}{2}+\frac{\pi}{4}\big)\big)W_{0,\star}(-2\mathrm{i}z)$ and $H^{(2)}_{\star}(z)=(2/\pi z)^{1/2}\,\exp\big(\mathrm{i}\big(\frac{\pi\star}{2}+\frac{\pi}{4}\big)\big)W_{0,\star}(2\mathrm{i}z)$; hence, as a consequence, one needs the small-argument asymptotic expansion of the Whittaker function:

$$W_{z_1,z_2}(z)\underset{\substack{z\to 0\\ |\arg z|<\pi}}{=} z^{1/2}\sum_{k=0}^{2}\sum_{l=\pm 1} w_k(z_1,lz_2)z^k+\mathcal{O}\Big(z^3 z^{-|\mathrm{Re}(z_2)|}\Big), \tag{131}$$

where

$$w_0(z_1,lz_2)=\frac{\Gamma(-2lz_2)z^{lz_2}}{\Gamma(\frac{1}{2}-lz_2-z_1)}, \qquad \frac{w_1(z_1,lz_2)}{w_0(z_1,lz_2)}=-\frac{1}{2}+\frac{(lz_2-z_1+\frac{1}{2})}{(2lz_2+1)},$$
$$\frac{w_2(z_1,lz_2)}{w_0(z_1,lz_2)}=\frac{1}{8}-\frac{(lz_2-z_1+\frac{1}{2})}{2(2lz_2+1)}+\frac{(lz_2-z_1+\frac{1}{2})(lz_2-z_1+\frac{3}{2})}{2(2lz_2+1)(2lz_2+2)}.$$

The latter expansion is obtained by using a representation of $W_{z_1,z_2}(\cdot)$ in terms of the confluent hypergeometric function [32], $\mathbf{\Phi}(\alpha,\beta;z):=\sum_{n=0}^{\infty}\frac{(\alpha)_n z^n}{(\beta)_n n!}$:

$$\begin{aligned}W_{z_1,z_2}(z)=\mathrm{e}^{-\frac{z}{2}}z^{\frac{1}{2}+z_2}\Bigg(&\frac{\Gamma(-2z_2)}{\Gamma(\frac{1}{2}-z_2-z_1)}\mathbf{\Phi}(z_2-z_1+\tfrac{1}{2},2z_2+1;z)+\frac{\Gamma(2z_2)}{\Gamma(\frac{1}{2}+z_2-z_1)}\\ &\times z^{-2z_2}\,\mathbf{\Phi}(-z_2-z_1+\tfrac{1}{2},-2z_2+1;z)\Bigg).\end{aligned} \tag{132}$$

Consider, for example, $g_{11}:=(G)_{11}$; the remaining matrix elements are estimated analogously. From Equations (130) and (131), and the duplication formula for the gamma function [31], $\Gamma(2x)=\frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma(x+\frac{1}{2})$, one shows that $g_{11}\underset{\substack{\tau\to+0\\ \epsilon_2<\lambda<\epsilon_1}}{=}((g_{11})_0+(g_{11})_{\Delta}+\Delta\mathcal{E}_{11})\big(1+o(\tau^{\delta_2})\big)$, $\delta_2>0$, where

$$\begin{aligned}(g_{11})_0 :=& \sum_{l=\pm 1}\frac{2^{l\widehat{\rho}}\mathrm{e}^{\frac{\mathrm{i}\pi l\widehat{\rho}}{2}}(\varepsilon b)^{-l\widehat{\rho}}(\Gamma(2l\widehat{\rho}))^2\tau^{-2l\widehat{\rho}}\mathrm{e}^{-\frac{\pi a}{4}}(2\tau)^{\frac{a\mathrm{i}}{2}}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}}{4\sqrt{\pi\varepsilon b}\,\Gamma\big(l\widehat{\rho}+\frac{a\mathrm{i}}{2}\big)}\\ &\times\left(\left(-4l\widehat{\rho}+2a\mathrm{i}+\frac{4B\,\widehat{\gamma}}{\sqrt{-AB}}\right)\left(-\sqrt{B}+\frac{\widehat{\delta}\sqrt{-AB}}{\sqrt{B}\,\big(l\widehat{\rho}+\frac{a\mathrm{i}}{2}\big)}\right)-\frac{2\mathrm{i}\varepsilon b\sqrt{B}}{\sqrt{-AB}}\right),\\ (g_{11})_{\Delta} :=& \sum_{l=\pm 1}\frac{2^{3l\widehat{\rho}}\mathrm{e}^{\frac{3\pi\mathrm{i}l\widehat{\rho}}{2}}(\varepsilon b)^{-l\widehat{\rho}}\lambda^{2l\widehat{\rho}}\Gamma(l\widehat{\rho})\Gamma(-l\widehat{\rho})\mathrm{e}^{-\frac{\pi a}{4}}(2\tau)^{\frac{a\mathrm{i}}{2}}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}}{4\sqrt{\pi\varepsilon b}\,\Gamma\big(-l\widehat{\rho}+\frac{a\mathrm{i}}{2}\big)}\\ &\times\left(\left(-4l\widehat{\rho}+2a\mathrm{i}+\frac{4B\,\widehat{\gamma}}{\sqrt{-AB}}\right)\left(-\sqrt{B}+\frac{\widehat{\delta}\sqrt{-AB}}{\sqrt{B}\,\big(-l\widehat{\rho}+\frac{a\mathrm{i}}{2}\big)}\right)-\frac{2\mathrm{i}\varepsilon b\sqrt{B}}{\sqrt{-AB}}\right),\\ \Delta\mathcal{E}_{11} :=& \tau^{2+\frac{a\mathrm{i}}{2}}\sum_{l_1=\tau,\lambda}\sum_{l_2=\pm 1} l_1^{2l_2\widehat{\rho}}\Bigg(a_0^{l_1}(l_2)\sqrt{B}+\frac{a_1^{l_1}(l_2)\widehat{\delta}\sqrt{-AB}}{\sqrt{B}}+\frac{a_2^{l_1}(l_2)B\sqrt{B}\,\widehat{\gamma}}{\sqrt{-AB}}\\ &+\frac{a_3^{l_1}(l_2)\sqrt{B}}{\sqrt{-AB}}\Bigg),\end{aligned}$$

with $a_k^{l_1}(l_2)$, $k=0,1,2,3$, $l_1=\tau,\lambda$, $l_2=\pm 1$, some $\mathcal{O}(1)$ coefficients (only the structure of $\Delta\mathcal{E}_{11}$ is essential). Recalling that $\widehat{\rho}^2:=\widehat{\gamma}\widehat{\delta}-a^2/4$ (Equation (111)), it follows from Equations (13) and (14) that $(g_{11})_{\Delta}=0$. Using the restrictions stated in the Lemma, one estimates $\Delta\mathcal{E}_{11}$ as $o(\tau^{\delta_1})$, $\delta_1>0$. Finally, Equation (126) is obtained after simplification of the expression for $(g_{11})_0$ upon using Equations (13) and (14), and the definition of $\widehat{\rho}^2$. □

**Lemma 5.2.** *Let* $0<\epsilon<1$,

$$a\neq 0, \qquad \textit{and} \qquad \widehat{\rho}=0\ \ (\equiv\nu_o=0).$$

*Then, under conditions* (113), (122), (123), *and*

$$\max\Big\{\Big|\sqrt{-A}\Big|,\Big|\widehat{\gamma}\sqrt{B}\Big|\Big\}\underset{\tau\to+0}{>}\tau^{1-\epsilon}\max\Big\{\Big|\sqrt{B}\Big|,\Big|\frac{\widehat{\gamma}}{\sqrt{-A}}\Big|,\Big|\frac{\widehat{\gamma}^2 B}{\sqrt{-A}}\Big|\Big\},$$

$$\max\Big\{\Big|\sqrt{B}\Big|,\Big|\widehat{\delta}\sqrt{-A}\Big|\Big\}\underset{\tau\to+0}{>}\tau^{1-\epsilon}\max\Bigg\{\Big|\sqrt{-A}\Big|,\frac{1}{\Big|\sqrt{-A}\Big|},\Big|\frac{B\widehat{\gamma}}{\sqrt{-A}}\Big|\Bigg\},$$

*or conditions* (113), (122), (124), *and*

$$\max\Big\{\Big|\sqrt{B}\Big|,\Big|\widehat{\delta}\sqrt{-A}\Big|\Big\}\underset{\tau\to+0}{>}\tau^{1-\epsilon}\max\Big\{\Big|\frac{\widehat{\delta}}{\sqrt{B}}\Big|,\Big|\frac{\widehat{\delta}^2 A}{\sqrt{B}}\Big|\Big\},$$

$$\max\Big\{\Big|\sqrt{-A}\Big|,\Big|\widehat{\gamma}\sqrt{B}\Big|\Big\}\underset{\tau\to+0}{>}\tau^{1-\epsilon}\max\Bigg\{\frac{1}{\Big|\sqrt{B}\Big|},\Big|\frac{\widehat{\delta}A}{\sqrt{B}}\Big|\Bigg\},$$

*the connection matrix, $G$ (Equation* (28)*), has the following asymptotics* ($g_{ij}:=(G)_{ij}$, $i,j=1,2$),

$$g_{11}\underset{\tau\to+0}{=}\frac{(2\tau)^{\frac{a\mathrm{i}}{2}}\mathrm{e}^{-\frac{\pi a}{4}}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}}{\sqrt{\pi\varepsilon b}\,\Gamma\big(\frac{a\mathrm{i}}{2}\big)}\Bigg(\Bigg(\sqrt{B}+\frac{2\mathrm{i}\widehat{\delta}\sqrt{-AB}}{a\sqrt{B}}\Bigg)\Bigg(4\psi(1)-\psi(\tfrac{a\mathrm{i}}{2})+\ln 2+\frac{\pi\mathrm{i}}{2}-\ln(\varepsilon b)$$
$$-\;2\ln\tau\Big)-\frac{4\widehat{\delta}\sqrt{-AB}}{a^2\sqrt{B}}\Bigg)\Big(1+o\Big(\tau^{\delta}\Big)\Big),\tag{133}$$

$$g_{12}\underset{\tau\to+0}{=}-\frac{(2\tau)^{-\frac{a\mathrm{i}}{2}}\mathrm{e}^{-\frac{\pi a}{4}}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}}{\sqrt{\pi\varepsilon b}\,\Gamma\big(-\frac{a\mathrm{i}}{2}\big)}\Bigg(\Bigg(\frac{\sqrt{-AB}}{\sqrt{B}}-\frac{2\mathrm{i}\widehat{\gamma}\sqrt{B}}{a}\Bigg)\Bigg(4\psi(1)-\psi(-\tfrac{a\mathrm{i}}{2})+\ln 2+\frac{3\pi\mathrm{i}}{2}-\ln(\varepsilon b)$$
$$-\;2\ln\tau\Big)-\frac{4\widehat{\gamma}\sqrt{B}}{a^2}\Bigg)\Big(1+o\Big(\tau^{\delta}\Big)\Big),\tag{134}$$

$$g_{21}\underset{\tau\to+0}{=}-\frac{(2\tau)^{\frac{a\mathrm{i}}{2}}\mathrm{e}^{-\frac{\pi a}{4}}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}}{\sqrt{\pi\varepsilon b}\,\Gamma\big(\frac{a\mathrm{i}}{2}\big)}\Bigg(\Bigg(\sqrt{B}+\frac{2\mathrm{i}\widehat{\delta}\sqrt{-AB}}{a\sqrt{B}}\Bigg)\Bigg(4\psi(1)-\psi(\tfrac{a\mathrm{i}}{2})+\ln 2-\frac{3\pi\mathrm{i}}{2}-\ln(\varepsilon b)$$
$$-\;2\ln\tau\Big)-\frac{4\widehat{\delta}\sqrt{-AB}}{a^2\sqrt{B}}\Bigg)\Big(1+o\Big(\tau^{\delta}\Big)\Big),\tag{135}$$

$$g_{22}\underset{\tau\to+0}{=}\frac{(2\tau)^{-\frac{a\mathrm{i}}{2}}\mathrm{e}^{-\frac{\pi a}{4}}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}}{\sqrt{\pi\varepsilon b}\,\Gamma\big(-\frac{a\mathrm{i}}{2}\big)}\Bigg(\Bigg(\frac{\sqrt{-AB}}{\sqrt{B}}-\frac{2\mathrm{i}\widehat{\gamma}\sqrt{B}}{a}\Bigg)\Bigg(4\psi(1)-\psi(-\tfrac{a\mathrm{i}}{2})+\ln 2-\frac{\pi\mathrm{i}}{2}-\ln(\varepsilon b)$$
$$-\;2\ln\tau\Big)-\frac{4\widehat{\gamma}\sqrt{B}}{a^2}\Bigg)\Big(1+o\Big(\tau^{\delta}\Big)\Big),\tag{136}$$

*where* $\psi(z):=\frac{\mathrm{d}}{\mathrm{d}z}\ln\Gamma(z)$ *is the psi function,* $\psi(1)=-0.57721566490\ldots$ [31], *and* $\delta>0$.

*Proof.* One proceeds as in the proof of Lemma 5.1; but, since $\widehat{\rho}=0$, the following representation [32] for the Whittaker function (instead of Equation (132)) is used,

$$W_{z_1,z_2}(z)=\frac{(-1)^{2z_2}z^{z_2+\frac{1}{2}}\mathrm{e}^{-\frac{z}{2}}}{\Gamma\big(\frac{1}{2}-z_2-z_1\big)\,\Gamma\big(\frac{1}{2}+z_2-z_1\big)}\Bigg(\sum_{k=0}^{\infty}\frac{\Gamma(z_2+k-z_1+\frac{1}{2})}{k!(2z_2+k)!}(\psi(k+1)+\psi(2z_2+k+1)$$
$$-\;\psi(z_2+k-z_1+\tfrac{1}{2})-\ln z)z^k+(-z)^{-2z_2}\sum_{k=0}^{2z_2-1}\frac{\Gamma(2z_2-k)\Gamma(k-z_2-z_1+\frac{1}{2})}{k!}(-z)^k\Bigg),$$

$|\arg z| < 3\pi/2$, $2z_2+1 \in \mathbb{N}$, and, when $z_2=0$, the second sum in the above expansion is equal to zero. The following identities for the Whittaker and psi functions were used [31]: $W_{0,-1}(\cdot)=W_{0,1}(\cdot)$, $\psi(z+1)=\psi(z)+\frac{1}{z}$, and $\psi(\frac{1}{2})=\psi(1)-2\ln 2$. □

**Proposition 5.5.** *Let $G$ be the connection matrix of Equation* (55) *with $\tau$-independent elements satisfying conditions* (43)*, and $\mathfrak{p}(z_1,z_2)$, $\chi_1(\vec{g};z_3) := \chi_1(\vec{g}(0,0);z_3)$ and $\chi_2(\vec{g};z_4) := \chi_2(\vec{g}(0,0);z_4)$, where $g_{ij}(0,0)=g_{ij}$, $i,j=1,2$, be defined as in Equations* (46) *and* (47)*. Then the corresponding isomonodromy deformations have the following asymptotic representation,*

$$\frac{\sqrt{-AB}}{\sqrt{B}} \underset{\tau\to+0}{=} \frac{\mathrm{i}\mathrm{e}^{\frac{\pi a}{4}}(2\tau)^{\frac{a\mathrm{i}}{2}}}{4\sqrt{\pi}\,(\varepsilon b)^{-1/2}}\Big(\mathfrak{p}(-a,\widehat{\rho})\mathrm{e}^{-\mathrm{i}\pi\widehat{\rho}}\chi_2(\vec{g};\widehat{\rho})\tau^{2\widehat{\rho}}+\mathfrak{p}(-a,-\widehat{\rho})\mathrm{e}^{\mathrm{i}\pi\widehat{\rho}}\chi_2(\vec{g};-\widehat{\rho})\tau^{-2\widehat{\rho}}\Big) \times\Big(1+o\Big(\tau^{\delta}\Big)\Big), \tag{137}$$

$$\sqrt{B}\,\widehat{\gamma} \underset{\tau\to+0}{=} \frac{\mathrm{i}\mathrm{e}^{\frac{\pi a}{4}}(2\tau)^{\frac{a\mathrm{i}}{2}}}{4\sqrt{\pi}\,(\varepsilon b)^{-1/2}}\bigg(\Big(-\widehat{\rho}-\frac{a\mathrm{i}}{2}\Big)\mathfrak{p}(-a,\widehat{\rho})\mathrm{e}^{-\mathrm{i}\pi\widehat{\rho}}\chi_2(\vec{g};\widehat{\rho})\tau^{2\widehat{\rho}}+\Big(\widehat{\rho}-\frac{a\mathrm{i}}{2}\Big)\mathfrak{p}(-a,-\widehat{\rho})\mathrm{e}^{\mathrm{i}\pi\widehat{\rho}} \times\chi_2(\vec{g};-\widehat{\rho})\tau^{-2\widehat{\rho}}\bigg)\Big(1+o\Big(\tau^{\delta}\Big)\Big), \tag{138}$$

$$\sqrt{B} \underset{\tau\to+0}{=} -\frac{\mathrm{i}\mathrm{e}^{\frac{\pi a}{4}}(2\tau)^{-\frac{a\mathrm{i}}{2}}}{4\sqrt{\pi}\,(\varepsilon b)^{-1/2}}\Big(\mathfrak{p}(a,\widehat{\rho})\chi_1(\vec{g};\widehat{\rho})\tau^{2\widehat{\rho}}+\mathfrak{p}(a,-\widehat{\rho})\chi_1(\vec{g};-\widehat{\rho})\tau^{-2\widehat{\rho}}\Big) \times\Big(1+o\Big(\tau^{\delta}\Big)\Big), \tag{139}$$

$$\frac{\widehat{\delta}\sqrt{-AB}}{\sqrt{B}} \underset{\tau\to+0}{=} -\frac{\mathrm{i}\mathrm{e}^{\frac{\pi a}{4}}(2\tau)^{-\frac{a\mathrm{i}}{2}}}{4\sqrt{\pi}\,(\varepsilon b)^{-1/2}}\bigg(\Big(-\widehat{\rho}+\frac{a\mathrm{i}}{2}\Big)\mathfrak{p}(a,\widehat{\rho})\chi_1(\vec{g};\widehat{\rho})\tau^{2\widehat{\rho}}+\Big(\widehat{\rho}+\frac{a\mathrm{i}}{2}\Big)\mathfrak{p}(a,-\widehat{\rho})\chi_1(\vec{g};-\widehat{\rho}) \times\tau^{-2\widehat{\rho}}\bigg)\Big(1+o\Big(\tau^{\delta}\Big)\Big), \tag{140}$$

$$\sqrt{-AB} \underset{\tau\to+0}{=} \frac{\varepsilon b\mathrm{e}^{\frac{\pi a}{2}}}{16\pi}\Big(\mathfrak{p}(a,\widehat{\rho})\chi_1(\vec{g};\widehat{\rho})\tau^{2\widehat{\rho}}+\mathfrak{p}(a,-\widehat{\rho})\chi_1(\vec{g};-\widehat{\rho})\tau^{-2\widehat{\rho}}\Big) \times\Big(\mathfrak{p}(-a,\widehat{\rho})\mathrm{e}^{-\mathrm{i}\pi\widehat{\rho}}\chi_2(\vec{g};\widehat{\rho})\tau^{2\widehat{\rho}}+\mathfrak{p}(-a,-\widehat{\rho})\mathrm{e}^{\mathrm{i}\pi\widehat{\rho}}\chi_2(\vec{g};-\widehat{\rho})\tau^{-2\widehat{\rho}}\Big)\Big(1+o\Big(\tau^{\delta}\Big)\Big), \tag{141}$$

$$\widehat{\gamma}:=\tau C \underset{\tau\to+0}{=} -\frac{\big((-\widehat{\rho}-\frac{a\mathrm{i}}{2})\mathfrak{p}(-a,\widehat{\rho})\mathrm{e}^{-\mathrm{i}\pi\widehat{\rho}}\chi_2(\vec{g};\widehat{\rho})\tau^{2\widehat{\rho}}+(\widehat{\rho}-\frac{a\mathrm{i}}{2})\mathfrak{p}(-a,-\widehat{\rho})\mathrm{e}^{\mathrm{i}\pi\widehat{\rho}}\chi_2(\vec{g};-\widehat{\rho})\tau^{-2\widehat{\rho}}\big)}{(2\tau)^{-a\mathrm{i}}\,\big(\mathfrak{p}(a,\widehat{\rho})\chi_1(\vec{g};\widehat{\rho})\tau^{2\widehat{\rho}}+\mathfrak{p}(a,-\widehat{\rho})\chi_1(\vec{g};-\widehat{\rho})\tau^{-2\widehat{\rho}}\big)} \times\Big(1+o\Big(\tau^{\delta}\Big)\Big), \tag{142}$$

$$\widehat{\delta}:=\tau D \underset{\tau\to+0}{=} -\frac{\big((-\widehat{\rho}+\frac{a\mathrm{i}}{2})\mathfrak{p}(a,\widehat{\rho})\chi_1(\vec{g};\widehat{\rho})\tau^{2\widehat{\rho}}+(\widehat{\rho}+\frac{a\mathrm{i}}{2})\mathfrak{p}(a,-\widehat{\rho})\chi_1(\vec{g};-\widehat{\rho})\tau^{-2\widehat{\rho}}\big)}{(2\tau)^{a\mathrm{i}}\,\big(\mathfrak{p}(-a,\widehat{\rho})\mathrm{e}^{-\mathrm{i}\pi\widehat{\rho}}\chi_2(\vec{g};\widehat{\rho})\tau^{2\widehat{\rho}}+\mathfrak{p}(-a,-\widehat{\rho})\mathrm{e}^{\mathrm{i}\pi\widehat{\rho}}\chi_2(\vec{g};-\widehat{\rho})\tau^{-2\widehat{\rho}}\big)} \times\Big(1+o\Big(\tau^{\delta}\Big)\Big), \tag{143}$$

*with $\delta>0$.*

*Proof.* Algebraically inverting Equations (126)–(129) and using the well-known identities for the gamma function [31] $\Gamma(2z)=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma(z+\frac{1}{2})$ and $\Gamma(\frac{1}{2}+z)\Gamma(\frac{1}{2}-z)=\frac{\pi}{\cos(\pi z)}$, one obtains Equations (137)–(143). It is easy to verify that the system of Equations (137)–(143) is compatible. Assuming that $g_{ij}$, $i,j=1,2$, are constant, $\widehat{\rho}\neq 0$, and $|\mathrm{Re}(\widehat{\rho})|<1/2$, one proves that any functions $A$, $B$, $C$, and $D$ with asymptotics (137)–(143) satisfy restrictions (113), (122), (123), (124), and those given in Lemma 5.1; therefore, one is now in a position to use the justification scheme suggested in [24] to complete the proof. □

**Proposition 5.6.** *Under the conditions of Proposition* 5.5*,*

$$\cos(2\pi\widehat{\rho}) \underset{\tau\to+0}{=} \cos(2\pi\rho)\Big(1+o\Big(\tau^{\delta}\Big)\Big), \tag{144}$$

*where $\cos(2\pi\rho)$ is defined in Equation* (44)*, and $\delta>0$.*

*Proof.* One recalls from Equations (111) that $\widehat{\rho}^2+a^2/4=\widehat{\gamma}\,\widehat{\delta}$. Substituting the asymptotic expressions for $\widehat{\gamma}$ and $\widehat{\delta}$ given in Equations (142) and (143) into this relation and simplifying, one shows that

$$\begin{aligned}&\left(\widehat{\rho}-\frac{a\mathrm{i}}{2}\right)\mathfrak{p}(a,\widehat{\rho})\mathfrak{p}(-a,-\widehat{\rho})\mathrm{e}^{\mathrm{i}\pi\widehat{\rho}}\chi_1(\vec{g};\widehat{\rho})\chi_2(\vec{g};-\widehat{\rho})\\&+\left(\widehat{\rho}+\frac{a\mathrm{i}}{2}\right)\mathfrak{p}(a,-\widehat{\rho})\mathfrak{p}(-a,\widehat{\rho})\mathrm{e}^{-\mathrm{i}\pi\widehat{\rho}}\chi_1(\vec{g};-\widehat{\rho})\chi_2(\vec{g};\widehat{\rho})\underset{\tau\to+0}{=}o\left(\tau^{\delta}\right),\quad\delta>0,\end{aligned}$$

with $\mathfrak{p}(z_1,z_2)$, $\chi_1(\vec{g};z_3):=\chi_1(\vec{g}(0,0);z_3)$ and $\chi_2(\vec{g};z_4):=\chi_2(\vec{g}(0,0);z_4)$, where $g_{ij}(0,0)=g_{ij}$, $i,j=1,2$, defined in Equations (46) and (47). Applying to the last formula the gamma function identity $\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}$, one arrives at

$$\begin{aligned}&\frac{\mathrm{e}^{\mathrm{i}\pi\widehat{\rho}}(\mathrm{i}g_{11}g_{12}-\mathrm{i}g_{21}g_{22}+g_{11}g_{22}\mathrm{e}^{2\pi\mathrm{i}\widehat{\rho}}+g_{12}g_{21}\mathrm{e}^{-2\pi\mathrm{i}\widehat{\rho}})}{\sin\left(\pi\left(\widehat{\rho}+\frac{a\mathrm{i}}{2}\right)\right)}\\&+\frac{\mathrm{e}^{-\mathrm{i}\pi\widehat{\rho}}(\mathrm{i}g_{11}g_{12}-\mathrm{i}g_{21}g_{22}+g_{11}g_{22}\mathrm{e}^{-2\pi\mathrm{i}\widehat{\rho}}+g_{12}g_{21}\mathrm{e}^{2\pi\mathrm{i}\widehat{\rho}})}{\sin\left(\pi\left(\widehat{\rho}-\frac{a\mathrm{i}}{2}\right)\right)}\underset{\tau\to+0}{=}o\left(\tau^{\delta}\right).\end{aligned}$$

Now, using trigonometric identities and the fact that $g_{11}g_{22}-g_{12}g_{21}=1$, one obtains, in the case $g_{11}g_{22}\neq 0$, Equation (144), where $\cos(2\pi\rho)$ is defined in Equation (44): the second relation of Equation (44) is a direct consequence of Equation (33). □

**Proposition 5.7.** *Let $G$ be the connection matrix of Equation* (55) *with $\tau$-independent elements satisfying conditions* (49), *and $Q(z)$, $\chi_1(\vec{g};z_3):=\chi_1(\vec{g}(0,0);z_3)$ and $\chi_2(\vec{g};z_4):=\chi_2(\vec{g}(0,0);z_4)$, where $g_{ij}(0,0)=g_{ij}$, $i,j=1,2$, be as defined in Equations* (51) *and* (47). *Then the corresponding isomonodromy deformations have the following asymptotic representation,*

$$\begin{aligned}\frac{\sqrt{-AB}}{\sqrt{B}}&\underset{\tau\to+0}{=}\frac{\mathrm{i}\sqrt{\varepsilon b}\,\Gamma\left(-\frac{a\mathrm{i}}{2}\right)\mathrm{e}^{\frac{\pi a}{4}}}{2\sqrt{\pi}\,(2\tau)^{-\frac{a\mathrm{i}}{2}}}\left(\chi_2(\vec{g};0)\left(1+\frac{a\mathrm{i}}{2}Q(-a)\right)+\frac{\pi a}{4}\left(g_{12}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}-3g_{22}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}\right)\right.\\&\left.-\,a\mathrm{i}\chi_2(\vec{g};0)\ln\tau\right)\left(1+o\left(\tau^{\delta}\right)\right),\end{aligned}\tag{145}$$

$$\begin{aligned}\sqrt{B}\,\widehat{\gamma}&\underset{\tau\to+0}{=}\frac{\mathrm{i}\sqrt{\varepsilon b}\,\Gamma\left(-\frac{a\mathrm{i}}{2}\right)a^2\mathrm{e}^{\frac{\pi a}{4}}}{8\sqrt{\pi}\,(2\tau)^{-\frac{a\mathrm{i}}{2}}}\left(\chi_2(\vec{g};0)Q(-a)+\frac{\mathrm{i}\pi}{2}\left(3g_{22}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}-g_{12}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\right)\right.\\&\left.-\,2\chi_2(\vec{g};0)\ln\tau\right)\left(1+o\left(\tau^{\delta}\right)\right),\end{aligned}\tag{146}$$

$$\begin{aligned}\sqrt{B}&\underset{\tau\to+0}{=}-\frac{\mathrm{i}\sqrt{\varepsilon b}\,\Gamma\left(\frac{a\mathrm{i}}{2}\right)\mathrm{e}^{\frac{\pi a}{4}}}{2\sqrt{\pi}\,(2\tau)^{\frac{a\mathrm{i}}{2}}}\left(\chi_1(\vec{g};0)\left(1-\frac{a\mathrm{i}}{2}Q(a)\right)+\frac{\pi a}{4}\left(g_{21}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}-3g_{11}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\right)\right.\\&\left.+\,a\mathrm{i}\chi_1(\vec{g};0)\ln\tau\right)\left(1+o\left(\tau^{\delta}\right)\right),\end{aligned}\tag{147}$$

$$\begin{aligned}\frac{\widehat{\delta}\sqrt{-AB}}{\sqrt{B}}&\underset{\tau\to+0}{=}-\frac{\mathrm{i}\sqrt{\varepsilon b}\,\Gamma\left(\frac{a\mathrm{i}}{2}\right)a^2\mathrm{e}^{\frac{\pi a}{4}}}{8\sqrt{\pi}\,(2\tau)^{\frac{a\mathrm{i}}{2}}}\left(\chi_1(\vec{g};0)Q(a)+\frac{\mathrm{i}\pi}{2}\left(g_{21}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}-3g_{11}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\right)\right.\\&\left.-\,2\chi_1(\vec{g};0)\ln\tau\right)\left(1+o\left(\tau^{\delta}\right)\right),\end{aligned}\tag{148}$$

$$\begin{aligned}\sqrt{-AB}&\underset{\tau\to+0}{=}\frac{\varepsilon b\mathrm{e}^{\frac{\pi a}{2}}}{2a\sinh\left(\frac{\pi a}{2}\right)}\left(\chi_1(\vec{g};0)\left(1-\frac{a\mathrm{i}}{2}Q(a)\right)+\frac{\pi a}{4}\left(g_{21}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}-3g_{11}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\right)+a\mathrm{i}\chi_1(\vec{g};0)\right.\\&\left.\times\,\ln\tau\right)\left(\chi_2(\vec{g};0)\left(1+\frac{a\mathrm{i}}{2}Q(-a)\right)+\frac{\pi a}{4}\left(g_{12}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}-3g_{22}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}\right)-a\mathrm{i}\chi_2(\vec{g};0)\ln\tau\right)\\&\times\left(1+o\left(\tau^{\delta}\right)\right),\end{aligned}\tag{149}$$

$$\widehat{\gamma}:=\tau C\underset{\tau\to+0}{=}-\frac{\pi a(2\tau)^{a\mathrm{i}}}{2\left(\Gamma\left(\frac{a\mathrm{i}}{2}\right)\right)^2\sinh\left(\frac{\pi a}{2}\right)}\frac{\left(\chi_2(\vec{g};0)Q(-a)+\frac{\mathrm{i}\pi}{2}(3g_{22}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}-g_{12}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}})-2\chi_2(\vec{g};0)\ln\tau\right)}{\left(\chi_1(\vec{g};0)(1-\frac{a\mathrm{i}}{2}Q(a))+\frac{\pi a}{4}(g_{21}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}-3g_{11}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}})+a\mathrm{i}\chi_1(\vec{g};0)\ln\tau\right)}$$

$$\times\left(1+o\left(\tau^{\delta}\right)\right), \tag{150}$$

$$\widehat{\delta}:=\tau D \underset{\tau\to+0}{=} -\frac{a^{3}\left(\Gamma\left(\frac{a\mathrm{i}}{2}\right)\right)^{2}\sinh\left(\frac{\pi a}{2}\right)}{8\pi(2\tau)^{a\mathrm{i}}}\frac{\left(\chi_{1}(\vec{g};0)Q(a)+\frac{\mathrm{i}\pi}{2}(g_{21}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}-3g_{11}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}})-2\chi_{1}(\vec{g};0)\ln\tau\right)}{\left(\chi_{2}(\vec{g};0)(1+\frac{a\mathrm{i}}{2}Q(-a))+\frac{\pi a}{4}(g_{12}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}-3g_{22}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}})-a\mathrm{i}\chi_{2}(\vec{g};0)\ln\tau\right)}$$

$$\times\left(1+o\left(\tau^{\delta}\right)\right), \tag{151}$$

*with* $\delta>0$.

*Proof.* Follows from Lemma 5.2 in a manner analogous to the proof of Proposition 5.5. □

Now, one can complete the proof (as $\tau\to+\infty$) of Theorems 3.4 and 3.5. The asymptotics for $u(\tau)$ given in Theorem 3.4 (resp., Theorem 3.5), Equation (45) (resp., Equation (50)) follows from Proposition 1.2 ($u(\tau)=\varepsilon\tau\sqrt{-AB}$, $\varepsilon=\pm1$) and Proposition 5.5, Equation (141) (resp., Proposition 5.7, Equation (149)). To get the asymptotics for $\mathcal{H}(\tau)$ given in these Theorems, one uses the second relation of Equation (37), where $u'(\tau)$ is calculated via

$$u'(\tau)=\frac{u(\tau)}{\tau}+2\varepsilon\tau(AD-BC),$$

and: (1) in the case $\widehat{\rho}\neq0$, Equation (44) and the following identities,

$$\mathfrak{p}(a,\rho)\mathfrak{p}(-a,-\rho)\mathrm{e}^{\mathrm{i}\pi\rho}+\frac{\pi\left(\rho+\frac{a\mathrm{i}}{2}\right)\mathrm{e}^{\mathrm{i}\pi\rho}}{\rho^{2}\sin\left(\pi\left(\rho+\frac{a\mathrm{i}}{2}\right)\right)}=0,$$
$$\frac{\mathrm{e}^{\mathrm{i}\pi\rho}\chi_{1}(\vec{g};\rho)\chi_{2}(\vec{g};-\rho)}{\sin\left(\pi\left(\rho+\frac{a\mathrm{i}}{2}\right)\right)}+\frac{\mathrm{e}^{-\mathrm{i}\pi\rho}\chi_{1}(\vec{g};-\rho)\chi_{2}(\vec{g};\rho)}{\sin\left(\pi\left(\rho-\frac{a\mathrm{i}}{2}\right)\right)}=0,$$
$$\frac{\mathrm{e}^{\mathrm{i}\pi\rho}\chi_{1}(\vec{g};\rho)\chi_{2}(\vec{g};-\rho)}{\sin\left(\pi\left(\rho+\frac{a\mathrm{i}}{2}\right)\right)}-\frac{\mathrm{e}^{-\mathrm{i}\pi\rho}\chi_{1}(\vec{g};-\rho)\chi_{2}(\vec{g};\rho)}{\sin\left(\pi\left(\rho-\frac{a\mathrm{i}}{2}\right)\right)}=4\mathrm{i}\mathrm{e}^{-\frac{\pi a}{2}};$$

and (2) in the case $\widehat{\rho}=0$, Equation (49) and

$$\psi(z+1)=\psi(z)+\frac{1}{z},\qquad \psi(1-z)=\psi(z)+\pi\cot(\pi z).$$

# 6 Transformations

In this section we collect Bäcklund transformations and Lie-point symmetries for Equation (1) and System (5). To apply these transformations for connection results, we consider their actions on solutions of Systems (4) and (12) ($\Phi$ and $\Psi$), and the manifold of monodromy data ($\mathcal{M}$).

## 6.1 Bäcklund Transformations

The Bäcklund transformations for System (5), as well as for Equation (1), can be constructed in a standard way via the Schlesinger transformations for the first equation of System (4) [11–13]. For the degenerate Painlevé V equation (equivalent to the "complete" third Painlevé equation), these transformations (with a parameter $\theta=\pm1$) were constructed in [18]; however, as noted in [18], these formulae are also applicable, without modification (simply set $\theta=0$), to the degenerate Painlevé III equation. For the reader's convenience, the latter formulae are given below. The set of Schlesinger transformations forms a group which acts covariantly on the set of solutions of System (4):

$$\Phi_{1}(\lambda,\tau)=\mathcal{R}\Phi(\lambda,\tau),$$

where $\Phi_{1}(\lambda,\tau)$ is the solution of System (4) with some functions $A_{1}(\tau)$, $B_{1}(\tau)$, $C_{1}(\tau)$, and $D_{1}(\tau)$, respectively, in place of $A(\tau)$, $B(\tau)$, $C(\tau)$, and $D(\tau)$ for $\Phi(\lambda,\tau)$. For System (4), the

group of Schlesinger transformations is a free cyclic group with generator $\mathcal{R}_{1,2}$[1]:

$$\mathcal{R}_{1,2}=\begin{pmatrix}0&0\\0&1\end{pmatrix}\left(\frac{\lambda}{\tau}\right)^{1/2}+\begin{pmatrix}1&\frac{A}{\sqrt{-AB}}\\-\frac{D}{2\mathrm{i}\tau}&-\frac{AD}{2\mathrm{i}\tau\sqrt{-AB}}\end{pmatrix}\left(\frac{\tau}{\lambda}\right)^{1/2},$$

$$A_1=\frac{2\mathrm{i}\tau}{D}\sqrt{-A_1B_1},\qquad B_1=-\frac{D}{2\mathrm{i}\tau}\sqrt{-A_1B_1},\qquad \sqrt{-A_1B_1}=-\frac{\mathrm{i}\varepsilon bD}{2\tau B},\qquad C_1=-\frac{2\mathrm{i}\tau A}{\sqrt{-AB}},$$

$$D_1=\frac{B}{2\mathrm{i}\tau}-\frac{D}{2\mathrm{i}\tau^2}\left(1+a\mathrm{i}+\frac{\tau AD}{\sqrt{-AB}}\right),\qquad \sqrt{-A_1B_1}=\sqrt{-AB}+\frac{1}{2\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\tau AD}{\sqrt{-AB}}\right),$$

$$\mathrm{i}a_1=\mathrm{i}a+1,\qquad u_1(\tau)=-\frac{\mathrm{i}\varepsilon b}{8u^2(\tau)}\big(\tau(-u'(\tau)+\mathrm{i}b)+(2a\mathrm{i}+1)u(\tau)\big).$$

The inverse transformation of $\mathcal{R}_{1,2}$ is

$$\mathcal{R}_{3,4}=\begin{pmatrix}1&0\\0&0\end{pmatrix}\left(\frac{\lambda}{\tau}\right)^{1/2}+\begin{pmatrix}-\frac{BC}{2\mathrm{i}\tau\sqrt{-AB}}&\frac{C}{2\mathrm{i}\tau}\\-\frac{B}{\sqrt{-AB}}&1\end{pmatrix}\left(\frac{\tau}{\lambda}\right)^{1/2},$$

$$A_1=-\frac{C}{2\mathrm{i}\tau}\sqrt{-A_1B_1},\qquad B_1=\frac{2\mathrm{i}\tau}{C}\sqrt{-A_1B_1},\qquad C_1=\frac{A}{2\mathrm{i}\tau}+\frac{C}{2\mathrm{i}\tau^2}\left(1-a\mathrm{i}-\frac{\tau BC}{\sqrt{-AB}}\right),$$

$$D_1=-\frac{2\mathrm{i}\tau B}{\sqrt{-AB}},\qquad \sqrt{-A_1B_1}=\sqrt{-AB}-\frac{1}{2\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\tau BC}{\sqrt{-AB}}\right),$$

$$\mathrm{i}a_1=\mathrm{i}a-1,\qquad u_1(\tau)=-\frac{\mathrm{i}\varepsilon b}{8u^2(\tau)}\big(\tau(u'(\tau)+\mathrm{i}b)+(2a\mathrm{i}-1)u(\tau)\big).$$

Denoting $v_n(\tau)=u_n(\tau)/\tau$, $n\in\mathbb{Z}$, where $u_n(\tau)$ is the general solution of Equation (1) corresponding to the coefficient $a:=a_0-\mathrm{i}n$, one arrives at the following differential-difference and difference equations:

$$\frac{\mathrm{i}\varepsilon b}{4}v_n'=v_n(v_{n+1}-v_{n-1}),\qquad v_n^2(v_{n+1}+v_{n-1})=\frac{\varepsilon b}{4\tau^2}(b+2(a_0-\mathrm{i}n)v_n).$$

The first one is the Kac-Moerbeke [33] equation, whilst the second should be equivalent to one of the so-called difference Painlevé equations. Consider the function $f_n(\tau)=p_n(\tau)q_n(\tau)/2$, where the functions $p_n(\tau)$ and $q_n(\tau)$ solve System (11) for $a=a_0-\mathrm{i}n$ and $\varepsilon_1=-1$. One finds that $f_n(\tau)=\frac{2\tau^2}{\mathrm{i}\varepsilon b}v_{n+1}(\tau)v_n(\tau):=\frac{2\tau^2}{\mathrm{i}\varepsilon b}g_n(\tau)$. Using, now, the above equations for the function $v_n(\tau)$, one proves that

$$2f_n(f_{n+1}+f_n+(\mathrm{i}a_0+n+1))(f_n+f_{n-1}+(\mathrm{i}a_0+n))=\mathrm{i}\varepsilon b\tau^2,$$

and

$$\left(\frac{\mathrm{i}\varepsilon b}{4}\right)^2\frac{\mathrm{d}^2\ln g_n}{\mathrm{d}\tau^2}=\Delta^2 g_n:=g_{n+1}+g_{n-1}-2g_n.$$

The last equation for $g_n=g_n(\tau)$ is a form of the Toda chain equation.

The double iteration of $\mathcal{R}_{1,2}$ (with abuse of notation, $\mathcal{R}_5=\mathcal{R}_{1,2}\circ\mathcal{R}_{1,2}$) reads

$$\mathcal{R}_5=\begin{pmatrix}0&0\\0&1\end{pmatrix}\frac{\lambda}{\tau}+\begin{pmatrix}0&\frac{2\mathrm{i}\tau}{D}\\-\frac{D}{2\mathrm{i}\tau}&\frac{\mathcal{X}}{2\mathrm{i}}\end{pmatrix},\qquad \mathcal{X}:=\frac{1+a\mathrm{i}}{\tau^2}-\frac{B}{\tau D},$$

$$\sqrt{-A_1B_1}=-\sqrt{-AB}+\frac{\tau B\mathcal{X}}{D},\qquad \sqrt{-A_1B_1}=\sqrt{-AB}+\frac{1}{2\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\tau B}{D}\right),$$

$$A_1=\frac{4\tau^2B}{D^2},\qquad B_1=(2\tau)^{-2}\left(AD^2-\tau\mathcal{X}(\tau B\mathcal{X}-2D\sqrt{-AB})\right),$$

[1]Note that there are two misprints in [18]: (i) the factor $DA$ which appears in the (2 2)-element of the right-most matrix of Equation (6.2) should be changed to $-DA$; and (ii) the function $\mathcal{W}$ which appears in Equation (6.4) should be changed to $\mathcal{W}=\frac{1+a\mathrm{i}}{\tau^2}-\frac{B}{\tau D}$.

$$C_1=\frac{4\tau^2}{D}, \qquad D_1=(2\tau)^{-2}\Big(CD^2+2D\sqrt{-AB}-(\tau B+D)\mathcal{X}\Big),$$
$$\mathrm{i}a_1=\mathrm{i}a+2.$$

The inverse of $\mathcal{R}_5$ is

$$\mathcal{R}_5=\begin{pmatrix}1&0\\0&0\end{pmatrix}\frac{\lambda}{\tau}+\begin{pmatrix}\frac{1}{4\mathrm{i}C}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{C}{\tau}\right)&\frac{C}{2\mathrm{i}\tau}\\-\frac{2\mathrm{i}\tau}{C}&0\end{pmatrix},$$
$$\sqrt{-A_1B_1}=\sqrt{-AB}-\frac{1}{2\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\tau A}{C}\right), \qquad \sqrt{-A_1B_1}=-\sqrt{-AB}+\frac{\tau A}{2C^2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{C}{\tau}\right),$$
$$A_1=(2\tau)^{-2}\Big(BC^2+\tau\mathcal{Y}(2C\sqrt{-AB}+\tau A\mathcal{Y})\Big), \qquad B_1=\frac{4\tau^2A}{C^2},$$
$$C_1=(2\tau)^{-2}\Big(C^2D+2C\sqrt{-AB}-(C-\tau A)\mathcal{Y}\Big), \qquad D_1=\frac{4\tau^2}{C},$$
$$\mathcal{Y}:=\frac{1-a\mathrm{i}}{\tau^2}+\frac{A}{\tau C}, \qquad \mathrm{i}a_1=\mathrm{i}a-2.$$

For the purpose of the application of these transformations to asymptotic results, one has to find the action of the group of Bäcklund transformations on the manifold of monodromy data. In terms of $\Psi(\mu,\tau)$ introduced in Proposition 2.1, the transformation $\mathcal{R}_{1,2}$ reads

$$\Psi_1(\mu,\tau)=\frac{\mu}{\sqrt{\tau}}\left(\begin{pmatrix}0&0\\0&1\end{pmatrix}+\frac{1}{\mu}\begin{pmatrix}0&\frac{2\mathrm{i}\tau}{D}\\-\frac{D}{2\mathrm{i}}&0\end{pmatrix}\right)\Psi(\mu,\tau):$$

applying the latter relation to the canonical solutions, one shows that

$$\mathcal{R}_{1,2}Y_k^{\infty}(\mu)=(Y_k^{\infty}(\mu))_1, \qquad \mathcal{R}_{1,2}X_k^0(\mu)=\mathrm{i}(X_k^0(\mu))_1\sigma_3;$$

hence,

$$(a,s_0^0,s_0^{\infty},s_1^{\infty},g_{11},g_{12},g_{21},g_{22})\to(a-\mathrm{i},-s_0^0,s_0^{\infty},s_1^{\infty},\mathrm{i}g_{11},\mathrm{i}g_{12},-\mathrm{i}g_{21},-\mathrm{i}g_{22}).$$

## 6.2 Lie-Point Symmetries

In this subsection, the subscripts "$n$" and "$o$" are used to denote "new" and "old" variables, respectively.

### 6.2.1 $\tau\to-\tau$

Let $A_o(\tau_o)$, $B_o(\tau_o)$, $C_o(\tau_o)$, and $D_o(\tau_o)$ be a solution of System (5), and the function $u_o(\tau_o)$ solve Equation (1). Set $\tau_n=\tau_o\mathrm{e}^{-\mathrm{i}\pi p}$, $p=\pm1$. Then

$$A_n(\tau_n)=-A_o(\tau_o), \quad B_n(\tau_n)=-B_o(\tau_o), \quad C_n(\tau_n)=C_o(\tau_o), \quad \text{and} \quad D_n(\tau_n)=D_o(\tau_o)$$

is a solution of System (5) for $\sqrt{-A_n(\tau_n)B_n(\tau_n)}=\sqrt{-A_o(\tau_o)B_o(\tau_o)}$, and the function $u_n(\tau_n)=-u_o(\tau_o)$ solves Equation (1). Note that $\widetilde{\alpha}_o(\tau_o)=\widetilde{\alpha}_n(\tau_n)$ and $-\mathrm{i}\widetilde{\alpha}_n(\tau_n)B_n(\tau_n)=\varepsilon b$, $\varepsilon=\pm1$. On the corresponding fundamental solutions of Systems (4) and (12), the above transformation acts as follows:

$$\lambda_o=-\lambda_n, \qquad \Phi_o(\lambda_o,\tau_o)=\sigma_3\Phi_n(\lambda_n,\tau_n)$$

and

$$\mu_o=\mu_n\mathrm{e}^{\frac{\mathrm{i}\pi l}{2}}, \quad l=\pm1, \qquad \Psi_o(\mu_o,\tau_o)=\mathrm{e}^{\frac{\mathrm{i}\pi l}{4}\sigma_3}\Psi_n(\mu_n,\tau_n).$$

In terms of the canonical solutions of System (12), this action reads:

$$Y_{o,k}^{\infty}(\mu_o)=\mathrm{e}^{\frac{\mathrm{i}\pi l}{4}\sigma_3}Y_{n,k-p-l}^{\infty}(\mu_n)\mathrm{e}^{-\frac{\mathrm{i}\pi l}{4}\sigma_3}\mathrm{e}^{\frac{\pi l}{2}(a-\mathrm{i}/2)\sigma_3},$$

$$X^0_{o,k}(\mu_o)=\begin{cases}\mathrm{e}^{\frac{\mathrm{i}\pi l}{4}\sigma_3}X^0_{n,k}(\mu_n), & p=l,\\ -\mathrm{i}l\mathrm{e}^{\frac{\mathrm{i}\pi l}{4}\sigma_3}X^0_{n,k-l}(\mu_n)\sigma_1, & p=-l.\end{cases}$$

These formulae for the canonical solutions imply the following action on $\mathcal{M}$:

$$S^\infty_{n,k-p-l}=\mathrm{e}^{-\frac{\mathrm{i}\pi l}{4}\sigma_3}\mathrm{e}^{\frac{\pi l}{2}(a-\mathrm{i}/2)\sigma_3}S^\infty_{o,k}\mathrm{e}^{-\frac{\pi l}{2}(a-\mathrm{i}/2)\sigma_3}\mathrm{e}^{\frac{\mathrm{i}\pi l}{4}\sigma_3},$$

$$S^0_{o,k}=\begin{cases}S^0_{n,k}, & p=l,\\ \sigma_1 S^0_{n,k-l}\sigma_1, & p=-l,\end{cases}$$

$$G_o=\begin{cases}-\mathrm{i}\sigma_1(S^0_{n,0})^{-1}G_n\mathrm{e}^{\frac{\mathrm{i}\pi}{4}\sigma_3}\mathrm{e}^{-\frac{\pi}{2}(a-\mathrm{i}/2)\sigma_3}, & p=1,\\ \mathrm{i}S^0_{n,0}\sigma_1 G_n\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}\sigma_3}\mathrm{e}^{\frac{\pi}{2}(a-\mathrm{i}/2)\sigma_3}, & p=-1.\end{cases}$$

Note that the action on $\mathcal{M}$ is independent of $l$.

One uses this transformation to prove the asymptotic results stated in Theorems 3.1–3.5 and formulate Conjectures 3.1 and 3.2 for $\varepsilon_1=\pm1$ (negative $\tau$) by using those for $\varepsilon_1=0$ (positive $\tau$).

### 6.2.2 $a\to -a$

Let $A_o(\tau_o)$, $B_o(\tau_o)$, $C_o(\tau_o)$, and $D_o(\tau_o)$ be a solution of System (5) for $a=a_o$, and the function $u_o(\tau_o)$ solve Equation (1) for $a=a_o$, $b=b_o$, and $\varepsilon=\varepsilon_o$ $(=\pm1)$. Set $a_n=-a_o$, and either $b_n=b_o$ and $\varepsilon_n=\varepsilon_o\mathrm{e}^{-\mathrm{i}\pi p}$, $p=\pm1$, or $b_n=b_o\mathrm{e}^{-\mathrm{i}\pi p}$ and $\varepsilon_n=\varepsilon_o$. Then

$$A_n(\tau_n)=-B_o(\tau_o),\quad B_n(\tau_n)=-A_o(\tau_o),\quad C_n(\tau_n)=D_o(\tau_o),\quad\text{and}\quad D_n(\tau_n)=C_o(\tau_o)$$

is a solution of System (5) for $a=-a_n$ and $\sqrt{-A_n(\tau_n)B_n(\tau_n)}=\sqrt{-A_o(\tau_o)B_o(\tau_o)}$, and the function $u_n(\tau_n)=\varepsilon_o\varepsilon_n u_o(\tau_o)$ solves Equation (1) for $a=-a_n$, $b=b_n$, and $\varepsilon=\varepsilon_n$. Note that $\widetilde{\alpha}_o(\tau_o)=\frac{B_n(\tau_n)}{A_n(\tau_n)}\widetilde{\alpha}_n(\tau_n)$ and $-\mathrm{i}\widetilde{\alpha}_n(\tau_n)B_n(\tau_n)=\varepsilon_n b_n$ $(\varepsilon_n=\pm1)$. On the corresponding fundamental solutions of Systems (4) and (12), the above transformation acts as follows:

$$\lambda_o=-\lambda_n,\qquad \Phi_o(\lambda_o,\tau_o)=\sigma_1\Phi_n(\lambda_n,\tau_n)$$

and

$$\mu_o=\mu_n\mathrm{e}^{\frac{\mathrm{i}\pi l}{2}},\quad l=\pm1,\qquad \Psi_o(\mu_o,\tau_o)=\mathcal{Q}(\mu_n,\tau_n)\Psi_n(\mu_n,\tau_n),$$

where

$$\mathcal{Q}(\mu_n,\tau_n):=\mu_n\mathrm{e}^{\frac{\mathrm{i}\pi l}{4}}\begin{pmatrix}0&0\\1&0\end{pmatrix}-\left(\frac{B_n(\tau_n)\mathrm{e}^{-\frac{\mathrm{i}\pi l}{4}}}{\sqrt{-A_n(\tau_n)B_n(\tau_n)}}\right)^{\sigma_3}\sigma_3.$$

In terms of the canonical solutions of System (12), this action reads:

$$Y^\infty_{o,k}(\mu_o)=\mathcal{Q}(\mu_n,\tau_n)Y^\infty_{n,k-l}(\mu_n)\mathrm{e}^{\frac{a_n\pi l}{2}\sigma_3}\sigma_1,$$

$$X^0_{o,k}(\mu_o)=\begin{cases}-\mathrm{i}p\mathcal{Q}(\mu_n,\tau_n)X^0_{n,k}(\mu_n), & p=l,\\ -\mathcal{Q}(\mu_n,\tau_n)X^0_{n,k+p}(\mu_n)\sigma_1, & p=-l.\end{cases}$$

These formulae for the canonical solutions imply the following action on $\mathcal{M}$:

$$S^\infty_{n,k-l}=\mathrm{e}^{\frac{a_n\pi l}{2}\sigma_3}\sigma_1 S^\infty_{o,k}\sigma_1\mathrm{e}^{-\frac{a_n\pi l}{2}\sigma_3},$$

$$S^0_{o,k}=\begin{cases}S^0_{n,k}, & p=l,\\ \sigma_1 S^0_{n,k+p}\sigma_1, & p=-l,\end{cases}$$

$$G_o=\begin{cases}\mathrm{i}G_n\mathrm{e}^{\pi(a_n-\mathrm{i}/2)\sigma_3}\sigma_3(S^\infty_{n,1})^{-1}\sigma_3\mathrm{e}^{-\pi(a_n-\mathrm{i}/2)\sigma_3}\mathrm{e}^{\frac{a_n\pi}{2}\sigma_3}\sigma_1, & p=1,\\ -S^0_{n,0}\sigma_1 G_n\mathrm{e}^{\pi(a_n-\mathrm{i}/2)\sigma_3}\sigma_3(S^\infty_{n,1})^{-1}\sigma_3\mathrm{e}^{-\pi(a_n-\mathrm{i}/2)\sigma_3}\mathrm{e}^{\frac{a_n\pi}{2}\sigma_3}\sigma_1, & p=-1.\end{cases}$$

Note that the action on $\mathcal{M}$ is independent of $l$.

One uses this transformation to prove the asymptotic results stated in Theorems 3.1–3.5 and formulate Conjectures 3.1 and 3.2 for $\varepsilon_2=\pm1$ (negative $\varepsilon b$) by using those for $\varepsilon_2=0$ (positive $\varepsilon b$).

### 6.2.3 $\tau \to \mathrm{i}\tau$

Let $A_o(\tau_o)$, $B_o(\tau_o)$, $C_o(\tau_o)$, and $D_o(\tau_o)$ be a solution of System (5), and the function $u_o(\tau_o)$ solve Equation (1) for $b=b_o$ and $\varepsilon=\varepsilon_o$. Set $\tau_n=-\mathrm{i}l\tau_o$, $l=\pm1$, and either $b_n=b_o$ and $\varepsilon_n=\varepsilon_o\mathrm{e}^{-\mathrm{i}\pi p}$, $p=\pm1$, or $b_n=b_o\mathrm{e}^{-\mathrm{i}\pi p}$ and $\varepsilon_n=\varepsilon_o$. Then

$$A_n(\tau_n)=-A_o(\tau_o),\quad B_n(\tau_n)=-B_o(\tau_o),\quad C_n(\tau_n)=\mathrm{i}lC_o(\tau_o),\quad \text{and}\quad D_n(\tau_n)=\mathrm{i}lD_o(\tau_o)$$

is a solution of System (5) for $\sqrt{-A_n(\tau_n)B_n(\tau_n)}=-\sqrt{-A_o(\tau_o)B_o(\tau_o)}$, and the function $u_n(\tau_n)=\mathrm{i}\varepsilon_o\varepsilon_n l u_o(\tau_o)$ solves Equation (1) for $b=b_n$ and $\varepsilon=\varepsilon_n$. Note that $\widetilde{\alpha}_o(\tau_o)=\widetilde{\alpha}_n(\tau_n)$ and $-\mathrm{i}\widetilde{\alpha}_n(\tau_n)B_n(\tau_n)=\varepsilon_n b_n$ ($\varepsilon_n=\pm1$). On the corresponding fundamental solutions of Systems (4) and (12), the above transformation acts as follows:

$$\lambda_o=-\mathrm{i}l\lambda_n,\qquad \Phi_o(\lambda_o,\tau_o)=\Phi_n(\lambda_n,\tau_n)$$

and

$$\mu_o=\mu_n\mathrm{e}^{-\frac{\mathrm{i}\pi l}{4}},\qquad \Psi_o(\mu_o,\tau_o)=\mathrm{e}^{\frac{\mathrm{i}\pi l}{8}\sigma_3}\Psi_n(\mu_n,\tau_n).$$

In terms of the canonical solutions of System (12), this action reads:

$$\begin{aligned}
Y^{\infty}_{o,k}(\mu_o)&=\mathrm{e}^{\frac{\mathrm{i}\pi l}{8}\sigma_3}Y^{\infty}_{n,k}(\mu_n)\mathrm{e}^{-\frac{\pi la}{4}\sigma_3},\\
X^0_{o,k}(\mu_o)&=\begin{cases}\mathrm{e}^{\frac{\mathrm{i}\pi l}{8}\sigma_3}X^0_{n,k}(\mu_n), & p=-l,\\ \mathrm{i}l\mathrm{e}^{\frac{\mathrm{i}\pi l}{8}\sigma_3}X^0_{n,k+l}(\mu_n)\sigma_1, & p=l.\end{cases}
\end{aligned}$$

These formulae for the canonical solutions imply the following action on $\mathcal{M}$:

$$\begin{aligned}
S^{\infty}_{n,k}&=\mathrm{e}^{-\frac{\pi la}{4}\sigma_3}S^{\infty}_{o,k}\mathrm{e}^{\frac{\pi la}{4}\sigma_3},\\
S^0_{o,k}&=\begin{cases}S^0_{n,k}, & p=-l,\\ \sigma_1 S^0_{n,k+l}\sigma_1, & p=l,\end{cases}\\
G_o&=\begin{cases}\mathrm{i}S^0_{n,0}\sigma_1 G_n\mathrm{e}^{\frac{\pi a}{4}\sigma_3}, & p=l=-1,\\ G_n\mathrm{e}^{-\frac{\pi a}{4}\sigma_3}, & p=-l=-1,\\ G_n\mathrm{e}^{\frac{\pi a}{4}\sigma_3}, & p=-l=1,\\ -\mathrm{i}\sigma_1(S^0_{n,0})^{-1}G_n\mathrm{e}^{-\frac{\pi a}{4}\sigma_3}, & p=l=1.\end{cases}
\end{aligned}$$

One uses this transformation to prove the asymptotic results stated in (see the Appendix) Theorems A.1–A.5 and formulate Conjectures A.1 and A.2 for (pure) imaginary $\tau$ by using the results for real $\tau$ in Section 3.

**Acknowledgements**

The authors are grateful to the Department of Pure Mathematics of the University of Adelaide, where part of this work was done, for hospitality. A. V. K. was partially supported by ARC grant # A69803721 and RFBR grant # 01-01-01045.

## Appendix: Asymptotics for Imaginary $\tau$

Here we present the summary of results for asymptotics of $u(\tau)$, $\mathcal{H}(\tau)$, and $\boldsymbol{\tau}(\tau)$ as $\tau\to\pm\mathrm{i}0$ and $\tau\to\pm\mathrm{i}\infty$. These results are obtained by applying the transformations changing $\tau\to\mathrm{i}\tau$ given in Subsection 6.2 to asymptotic results for real values of $\tau$ presented in Section 3.

In order to present these results, it is convenient to introduce the mapping $\widehat{\mathcal{F}}_{\varepsilon_1,\varepsilon_2}\colon\mathcal{M}\to\mathcal{M}$, $(a,s_0^0,s_0^{\infty},s_1^{\infty},g_{11},g_{12},g_{21},g_{22})\to(a,s_0^0,\widehat{s}_0^{\infty}(\varepsilon_1,\varepsilon_2),\widehat{s}_1^{\infty}(\varepsilon_1,\varepsilon_2),\widehat{g}_{11}(\varepsilon_1,\varepsilon_2),\widehat{g}_{12}(\varepsilon_1,\varepsilon_2),\widehat{g}_{21}(\varepsilon_1,\varepsilon_2),$ $\widehat{g}_{22}(\varepsilon_1,\varepsilon_2))$, $\varepsilon_1=\pm1$, $\varepsilon_2=0,\pm1$. Define:

(1) $\widehat{\mathcal{F}}_{-1,0}$ as: $\widehat{s}_0^{\infty}(-1,0)=s_1^{\infty}\mathrm{e}^{-\frac{3\pi a}{2}}$, $\widehat{s}_1^{\infty}(-1,0)=s_0^{\infty}\mathrm{e}^{-\frac{\pi a}{2}}$, $\widehat{g}_{11}(-1,0)=-g_{22}\mathrm{e}^{-\frac{3\pi a}{4}}$, $\widehat{g}_{12}(-1,0)=-(g_{21}+s_0^{\infty}g_{22})\mathrm{e}^{\frac{3\pi a}{4}}$, $\widehat{g}_{21}(-1,0)=-(g_{12}-s_0^0g_{22})\mathrm{e}^{-\frac{3\pi a}{4}}$, and $\widehat{g}_{22}(-1,0)=-(g_{11}+s_0^{\infty}g_{12}-(g_{21}+s_0^{\infty}g_{22})s_0^0)\mathrm{e}^{\frac{3\pi a}{4}}$;

(2) $\widehat{\mathcal{F}}_{-1,-1}$ as: $\widehat{s}_0^{\infty}(-1,-1)=s_0^{\infty}\mathrm{e}^{-\frac{\pi a}{2}}$, $\widehat{s}_1^{\infty}(-1,-1)=s_1^{\infty}\mathrm{e}^{\frac{\pi a}{2}}$, $\widehat{g}_{11}(-1,-1)=-\mathrm{i}g_{21}\mathrm{e}^{-\frac{\pi a}{4}}$, $\widehat{g}_{12}(-1,-1)=-\mathrm{i}g_{22}\mathrm{e}^{\frac{\pi a}{4}}$, $\widehat{g}_{21}(-1,-1)=-\mathrm{i}(g_{11}-s_0^0g_{21})\mathrm{e}^{-\frac{\pi a}{4}}$, and $\widehat{g}_{22}(-1,-1)=-\mathrm{i}(g_{12}-s_0^0g_{22})\mathrm{e}^{\frac{\pi a}{4}}$;

(3) $\widehat{\mathcal{F}}_{-1,1}$ as: $\widehat{s}_0^{\infty}(-1,1)=s_0^{\infty}\mathrm{e}^{-\frac{\pi a}{2}}$, $\widehat{s}_1^{\infty}(-1,1)=s_1^{\infty}\mathrm{e}^{\frac{\pi a}{2}}$, $\widehat{g}_{11}(-1,1)=g_{11}\mathrm{e}^{-\frac{\pi a}{4}}$, $\widehat{g}_{12}(-1,1)=g_{12}\mathrm{e}^{\frac{\pi a}{4}}$, $\widehat{g}_{21}(-1,1)=g_{21}\mathrm{e}^{-\frac{\pi a}{4}}$, and $\widehat{g}_{22}(-1,1)=g_{22}\mathrm{e}^{\frac{\pi a}{4}}$;

(4) $\widehat{\mathcal{F}}_{1,0}$ as: $\widehat{s}_0^{\infty}(1,0)=s_1^{\infty}\mathrm{e}^{-\frac{\pi a}{2}}$, $\widehat{s}_1^{\infty}(1,0)=s_0^{\infty}\mathrm{e}^{-\frac{3\pi a}{2}}$, $\widehat{g}_{11}(1,0)=-\mathrm{i}g_{12}\mathrm{e}^{-\frac{\pi a}{4}}$, $\widehat{g}_{12}(1,0)=-\mathrm{i}(g_{11}+s_0^{\infty}g_{12})\mathrm{e}^{\frac{\pi a}{4}}$, $\widehat{g}_{21}(1,0)=-\mathrm{i}g_{22}\mathrm{e}^{-\frac{\pi a}{4}}$, and $\widehat{g}_{22}(1,0)=-\mathrm{i}(g_{21}+s_0^{\infty}g_{22})\mathrm{e}^{\frac{\pi a}{4}}$;

(5) $\widehat{\mathcal{F}}_{1,-1}$ as: $\widehat{s}_0^{\infty}(1,-1)=s_0^{\infty}\mathrm{e}^{\frac{\pi a}{2}}$, $\widehat{s}_1^{\infty}(1,-1)=s_1^{\infty}\mathrm{e}^{-\frac{\pi a}{2}}$, $\widehat{g}_{11}(1,-1)=g_{11}\mathrm{e}^{\frac{\pi a}{4}}$, $\widehat{g}_{12}(1,-1)=g_{12}\mathrm{e}^{-\frac{\pi a}{4}}$, $\widehat{g}_{21}(1,-1)=g_{21}\mathrm{e}^{\frac{\pi a}{4}}$, and $\widehat{g}_{22}(1,-1)=g_{22}\mathrm{e}^{-\frac{\pi a}{4}}$; and

(6) $\widehat{\mathcal{F}}_{1,1}$ as: $\widehat{s}_0^{\infty}(1,1)=s_0^{\infty}\mathrm{e}^{\frac{\pi a}{2}}$, $\widehat{s}_1^{\infty}(1,1)=s_1^{\infty}\mathrm{e}^{-\frac{\pi a}{2}}$, $\widehat{g}_{11}(1,1)=\mathrm{i}(g_{21}+s_0^0g_{11})\mathrm{e}^{\frac{\pi a}{4}}$, $\widehat{g}_{12}(1,1)=\mathrm{i}(g_{22}+s_0^0g_{12})\mathrm{e}^{-\frac{\pi a}{4}}$, $\widehat{g}_{21}(1,1)=\mathrm{i}g_{11}\mathrm{e}^{\frac{\pi a}{4}}$, and $\widehat{g}_{22}(1,1)=\mathrm{i}g_{12}\mathrm{e}^{-\frac{\pi a}{4}}$.

**Theorem A.1.** *Let $\varepsilon_1=\pm1$, $\varepsilon_2=0,\pm1$, $\varepsilon b=|\varepsilon b|\mathrm{e}^{\mathrm{i}\pi\varepsilon_2}$, and $u(\tau)$ be a solution of Equation* (1) *corresponding to the monodromy data $(a,s_0^0,s_0^{\infty},s_1^{\infty},g_{11},g_{12},g_{21},g_{22})$. Suppose that*

$$\widehat{g}_{11}(\varepsilon_1,\varepsilon_2)\widehat{g}_{12}(\varepsilon_1,\varepsilon_2)\widehat{g}_{21}(\varepsilon_1,\varepsilon_2)\widehat{g}_{22}(\varepsilon_1,\varepsilon_2)\neq0,\qquad \left|\mathrm{Re}\left(\tfrac{\mathrm{i}}{2\pi}\ln(\widehat{g}_{11}(\varepsilon_1,\varepsilon_2)\widehat{g}_{22}(\varepsilon_1,\varepsilon_2))\right)\right|<\frac{1}{6}.$$

*Then $\exists\,\delta>0$ such that $u(\tau)$ has the asymptotic expansion*

$$\begin{aligned}u(\tau)\underset{\tau\to\infty\mathrm{e}^{\frac{\mathrm{i}\pi\varepsilon_1}{2}}}{=}&\frac{(-1)^{\frac{1+\varepsilon_1}{2}}\mathrm{i}\varepsilon\sqrt{|\varepsilon b|}}{3^{1/4}}\left(\sqrt{\frac{\vartheta(\tau)}{12}}+\sqrt{\widehat{\nu}(\varepsilon_1,\varepsilon_2)+1}\,\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}\cosh(\mathrm{i}\vartheta(\tau)+(\widehat{\nu}(\varepsilon_1,\varepsilon_2)+1)\right.\\&\left.\times\,\ln\vartheta(\tau)+\widehat{z}(\varepsilon_1,\varepsilon_2)+o\left(\tau^{-\delta}\right)\right)\Big),\end{aligned}\tag{A.1}$$

*where*

$$\vartheta(\tau):=3\sqrt{3}\,|\varepsilon b|^{1/3}|\tau|^{2/3},\qquad \widehat{\nu}(\varepsilon_1,\varepsilon_2)+1:=\frac{\mathrm{i}}{2\pi}\ln(\widehat{g}_{11}(\varepsilon_1,\varepsilon_2)\widehat{g}_{22}(\varepsilon_1,\varepsilon_2)),$$

$$\begin{aligned}\widehat{z}(\varepsilon_1,\varepsilon_2):=&\frac{1}{2}\ln(2\pi)-\frac{\pi\mathrm{i}}{2}-\frac{3\pi\mathrm{i}}{2}(\widehat{\nu}(\varepsilon_1,\varepsilon_2)+1)+(-1)^{\varepsilon_2}\mathrm{i}a\ln\left(2+\sqrt{3}\right)+(\widehat{\nu}(\varepsilon_1,\varepsilon_2)+1)\ln 12\\&-\ln\left(\widehat{\omega}(\varepsilon_1,\varepsilon_2)\sqrt{\widehat{\nu}(\varepsilon_1,\varepsilon_2)+1}\,\Gamma(\widehat{\nu}(\varepsilon_1,\varepsilon_2)+1)\right),\end{aligned}$$

*with*

$$\widehat{\omega}(\varepsilon_1,\varepsilon_2):=\frac{\widehat{g}_{12}(\varepsilon_1,\varepsilon_2)}{\widehat{g}_{22}(\varepsilon_1,\varepsilon_2)},$$

*and* $\Gamma(\cdot)$ *is the gamma function.*

*Let* $\mathcal{H}(\tau)$ *be the Hamiltonian function defined in Equation* (35) *corresponding to the function* $u(\tau)$ *given above. Then*

$$\mathcal{H}(\tau) \underset{\tau\to\infty e^{\frac{i\pi\varepsilon_1}{2}}}{=} (-1)^{\frac{1+\varepsilon_1}{2}} i\Big(3(\varepsilon b)^{2/3}|\tau|^{1/3}+2|\varepsilon b|^{1/3}|\tau|^{-1/3}\Big((a-(-1)^{\varepsilon_2}i/2)-2\sqrt{3}\,i(\widehat{\nu}(\varepsilon_1,\varepsilon_2)+1)$$
$$+\ o\Big(\tau^{-\delta}\Big)\Big)+\frac{(a-(-1)^{\varepsilon_2}i/2)^2}{2\tau}\Big). \tag{A.2}$$

**Theorem A.2.** *Let* $\varepsilon_1=\pm1$, $\varepsilon_2=0,\pm1$, $\varepsilon b=|\varepsilon b|e^{i\pi\varepsilon_2}$, *and* $u(\tau)$ *be a solution of Equation* (1) *corresponding to the monodromy data* $(a,s_0^0,s_0^\infty,s_1^\infty,g_{11},g_{12},g_{21},g_{22})$. *Suppose that*

$$\widehat{g}_{21}(\varepsilon_1,\varepsilon_2)=0, \qquad \widehat{g}_{11}(\varepsilon_1,\varepsilon_2)\widehat{g}_{22}(\varepsilon_1,\varepsilon_2)=1.$$

*Then* $\exists\ \delta>0$ *such that* $u(\tau)$ *has the asymptotic expansion*

$$u(\tau) \underset{\tau\to\infty e^{\frac{i\pi\varepsilon_1}{2}}}{=} (-1)^{\frac{1+\varepsilon_1}{2}} i\left(\frac{\varepsilon(\varepsilon b)^{2/3}}{2}|\tau|^{1/3}+\frac{(-1)^{\varepsilon_1}\varepsilon\sqrt{|\varepsilon b|}\,(s_0^0-ie^{(-1)^{\varepsilon_2+1}\pi a})}{2^{3/2}3^{1/4}\sqrt{\pi}}\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)^{(-1)^{\varepsilon_2}ia}\right.$$
$$\left.\times\ \exp\Big(-i\Big(3\sqrt{3}\,|\varepsilon b|^{1/3}|\tau|^{2/3}-\frac{\pi}{4}\Big)\Big)\right)\Big(1+o\Big(\tau^{-\delta}\Big)\Big). \tag{A.3}$$

**Theorem A.3.** *Let* $\varepsilon_1=\pm1$, $\varepsilon_2=0,\pm1$, $\varepsilon b=|\varepsilon b|e^{i\pi\varepsilon_2}$, *and* $u(\tau)$ *be a solution of Equation* (1) *corresponding to the monodromy data* $(a,s_0^0,s_0^\infty,s_1^\infty,g_{11},g_{12},g_{21},g_{22})$. *Suppose that*

$$\widehat{g}_{12}(\varepsilon_1,\varepsilon_2)=0, \qquad \widehat{g}_{11}(\varepsilon_1,\varepsilon_2)\widehat{g}_{22}(\varepsilon_1,\varepsilon_2)=1.$$

*Then* $\exists\ \delta>0$ *such that* $u(\tau)$ *has the asymptotic expansion*

$$u(\tau) \underset{\tau\to\infty e^{\frac{i\pi\varepsilon_1}{2}}}{=} (-1)^{\frac{1+\varepsilon_1}{2}} i\left(\frac{\varepsilon(\varepsilon b)^{2/3}}{2}|\tau|^{1/3}+\frac{(-1)^{\varepsilon_1}\varepsilon\sqrt{|\varepsilon b|}\,(s_0^0-ie^{(-1)^{\varepsilon_2+1}\pi a})}{2^{3/2}3^{1/4}\sqrt{\pi}}\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)^{(-1)^{\varepsilon_2}ia}\right.$$
$$\left.\times\ \exp\Big(i\Big(3\sqrt{3}\,|\varepsilon b|^{1/3}|\tau|^{2/3}+\frac{3\pi}{4}\Big)\Big)\right)\Big(1+o\Big(\tau^{-\delta}\Big)\Big). \tag{A.4}$$

**Theorem A.4.** *Let* $\varepsilon_1=\pm1$, $\varepsilon_2=0,\pm1$, $\varepsilon b=|\varepsilon b|e^{i\pi\varepsilon_2}$, *and* $u(\tau)$ *be a solution of Equation* (1) *corresponding to the monodromy data* $(a,s_0^0,s_0^\infty,s_1^\infty,g_{11},g_{12},g_{21},g_{22})$. *Suppose that*

$$|\mathrm{Im}(a)|<1, \qquad \widehat{g}_{11}(\varepsilon_1,\varepsilon_2)\widehat{g}_{22}(\varepsilon_1,\varepsilon_2)\neq0, \qquad \rho\neq0, \qquad |\mathrm{Re}(\rho)|<\frac{1}{2},$$

*where*

$$\cos(2\pi\rho):=-\frac{is_0^0}{2}=\cosh(\pi a)+\frac{1}{2}s_0^\infty s_1^\infty e^{\pi a}.$$

*Then* $\exists\ \delta>0$ *such that* $u(\tau)$ *has the asymptotic expansion*

$$u(\tau) \underset{\tau\to0e^{\frac{i\pi\varepsilon_1}{2}}}{=} \frac{(-1)^{\varepsilon_2}\tau b}{16\pi}\exp\Big((-1)^{\varepsilon_2}\frac{\pi a}{2}\Big)\Big(\mathfrak{p}((-1)^{\varepsilon_2}a,\rho)\chi_1(\vec{g}(\varepsilon_1,\varepsilon_2);\rho)|\tau|^{2\rho}+\mathfrak{p}((-1)^{\varepsilon_2}a,-\rho)$$
$$\times\ \chi_1(\vec{g}(\varepsilon_1,\varepsilon_2);-\rho)|\tau|^{-2\rho}\Big)\Big(\mathfrak{p}((-1)^{\varepsilon_2+1}a,\rho)e^{-i\pi\rho}\chi_2(\vec{g}(\varepsilon_1,\varepsilon_2);\rho)|\tau|^{2\rho}$$
$$+\ \mathfrak{p}((-1)^{\varepsilon_2+1}a,-\rho)e^{i\pi\rho}\chi_2(\vec{g}(\varepsilon_1,\varepsilon_2);-\rho)|\tau|^{-2\rho}\Big)\Big(1+\mathcal{O}\Big(\tau^{\delta}\Big)\Big), \tag{A.5}$$

*where*

$$\mathfrak{p}(z_1,z_2):=\left(\frac{|\varepsilon b|}{32}e^{\frac{i\pi}{2}}\right)^{z_2}\left(\frac{\Gamma(\frac{1}{2}-z_2)}{\Gamma(1+z_2)}\right)^2\frac{\Gamma(1+z_2+\frac{iz_1}{2})}{\tan(\pi z_2)},$$

$$\begin{aligned}
&\chi_1(\vec{\widehat{g}}(\varepsilon_1,\varepsilon_2);z_3):=\widehat{g}_{11}(\varepsilon_1,\varepsilon_2)\mathrm{e}^{\mathrm{i}\pi z_3}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}+\widehat{g}_{21}(\varepsilon_1,\varepsilon_2)\mathrm{e}^{-\mathrm{i}\pi z_3}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}},\\
&\chi_2(\vec{\widehat{g}}(\varepsilon_1,\varepsilon_2);z_4):=\widehat{g}_{12}(\varepsilon_1,\varepsilon_2)\mathrm{e}^{\mathrm{i}\pi z_4}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}+\widehat{g}_{22}(\varepsilon_1,\varepsilon_2)\mathrm{e}^{-\mathrm{i}\pi z_4}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}.
\end{aligned}\tag{A.6}$$

*Let $\mathcal{H}(\tau)$ be the Hamiltonian function defined in Equation* (35) *corresponding to the function $u(\tau)$ given above. Then*

$$\begin{aligned}
\mathcal{H}(\tau) \underset{\tau\to 0\mathrm{e}^{\frac{\mathrm{i}\pi\varepsilon_1}{2}}}{=} &\frac{2\rho}{\tau}\frac{\left(\mathfrak{p}((-1)^{\varepsilon_2}a,\rho)\chi_1(\vec{\widehat{g}}(\varepsilon_1,\varepsilon_2);\rho)|\tau|^{2\rho}-\mathfrak{p}((-1)^{\varepsilon_2}a,-\rho)\chi_1(\vec{\widehat{g}}(\varepsilon_1,\varepsilon_2);-\rho)|\tau|^{-2\rho}\right)}{\left(\mathfrak{p}((-1)^{\varepsilon_2},\rho)\chi_1(\vec{\widehat{g}}(\varepsilon_1,\varepsilon_2);\rho)|\tau|^{2\rho}+\mathfrak{p}((-1)^{\varepsilon_2}a,-\rho)\chi_1(\vec{\widehat{g}}(\varepsilon_1,\varepsilon_2);-\rho)|\tau|^{-2\rho}\right)}\\
&+\frac{1}{2\tau}\left(a(a-(-1)^{\varepsilon_2}\mathrm{i})+\frac{1}{4}+8\rho^2\right)+o\left(\frac{1}{\tau}\right).
\end{aligned}\tag{A.7}$$

**Theorem A.5.** *Let $\varepsilon_1=\pm1$, $\varepsilon_2=0,\pm1$, $\varepsilon b=|\varepsilon b|\mathrm{e}^{\mathrm{i}\pi\varepsilon_2}$, and $u(\tau)$ be a solution of Equation* (1) *corresponding to the monodromy data $(a,s_0^0,s_0^\infty,s_1^\infty,g_{11},g_{12},g_{21},g_{22})$. Suppose that*

$$|\mathrm{Im}(a)|<1,\qquad \widehat{g}_{11}(\varepsilon_1,\varepsilon_2)\widehat{g}_{22}(\varepsilon_1,\varepsilon_2)\neq 0,\qquad s_0^0=2\mathrm{i}.$$

*Then $\exists\ \delta>0$ such that $u(\tau)$ has the asymptotic expansion*

$$\begin{aligned}
u(\tau) \underset{\tau\to 0\mathrm{e}^{\frac{\mathrm{i}\pi\varepsilon_1}{2}}}{=} &\frac{(-1)^{\varepsilon_2}\tau b\exp((-1)^{\varepsilon_2}\frac{\pi a}{2})}{2a\sinh(\frac{\pi a}{2})}\Bigg(\chi_1(\vec{\widehat{g}}(\varepsilon_1,\varepsilon_2);0)\left(1-\tfrac{(-1)^{\varepsilon_2}\mathrm{i}a}{2}Q((-1)^{\varepsilon_2}a)\right)+\frac{(-1)^{\varepsilon_2}\pi a}{4}\\
&\times\ (\widehat{g}_{21}(\varepsilon_1,\varepsilon_2)\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}-3\widehat{g}_{11}(\varepsilon_1,\varepsilon_2)\mathrm{e}^{\frac{\mathrm{i}\pi}{4}})+(-1)^{\varepsilon_2}\mathrm{i}a\chi_1(\vec{\widehat{g}}(\varepsilon_1,\varepsilon_2);0)\ln|\tau|\Bigg)\\
&\times\ \Bigg(\chi_2(\vec{\widehat{g}}(\varepsilon_1,\varepsilon_2);0)\left(1+\tfrac{(-1)^{\varepsilon_2}\mathrm{i}a}{2}Q((-1)^{\varepsilon_2+1}a)\right)+\frac{(-1)^{\varepsilon_2}\pi a}{4}(\widehat{g}_{12}(\varepsilon_1,\varepsilon_2)\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\\
&-\ 3\widehat{g}_{22}(\varepsilon_1,\varepsilon_2)\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}})-(-1)^{\varepsilon_2}\mathrm{i}a\chi_2(\vec{\widehat{g}}(\varepsilon_1,\varepsilon_2);0)\ln|\tau|\Bigg)\left(1+\mathcal{O}\left(\tau^\delta\right)\right),
\end{aligned}\tag{A.8}$$

*where $\chi_j(\vec{\widehat{g}}(\varepsilon_1,\varepsilon_2);\cdot)$, $j=1,2$, are defined in Theorem* A.4*, Equations* (A.6)*,*

$$Q(z):=4\psi(1)-\psi(\mathrm{i}z/2)+\ln 2-\ln(|\varepsilon b|),$$

*$\psi(x):=\frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x)$ is the psi function, and $\psi(1)=-0.57721566490\ldots$.*

*Let $\mathcal{H}(\tau)$ be the Hamiltonian function defined in Equation* (35) *corresponding to the function $u(\tau)$ given above. Then*

$$\mathcal{H}(\tau) \underset{\tau\to 0\mathrm{e}^{\frac{\mathrm{i}\pi\varepsilon_1}{2}}}{=} \frac{1}{2\tau}\left(a(a-(-1)^{\varepsilon_2}\mathrm{i})+\frac{1}{4}\right)+\frac{\widehat{b}_2(\varepsilon_1,\varepsilon_2)}{\tau(\widehat{a}_2(\varepsilon_1,\varepsilon_2)+\widehat{b}_2(\varepsilon_1,\varepsilon_2)\ln|\tau|)}+o\left(\frac{1}{\tau}\right),\tag{A.9}$$

*where*

$$\begin{aligned}
\widehat{a}_2(\varepsilon_1,\varepsilon_2) :=&\ \chi_1(\vec{\widehat{g}}(\varepsilon_1,\varepsilon_2);0)\left(1-\tfrac{(-1)^{\varepsilon_2}\mathrm{i}a}{2}Q((-1)^{\varepsilon_2}a)\right)+\frac{(-1)^{\varepsilon_2}\pi a}{4}(\widehat{g}_{21}(\varepsilon_1,\varepsilon_2)\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}\\
&-3\widehat{g}_{11}(\varepsilon_1,\varepsilon_2)\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}),
\end{aligned}$$

$$\widehat{b}_2(\varepsilon_1,\varepsilon_2):=(-1)^{\varepsilon_2}\mathrm{i}a\chi_1(\vec{\widehat{g}}(\varepsilon_1,\varepsilon_2);0).$$

**Conjecture A.1.** *For the conditions stated in Theorem* A.4*,*

$$\begin{aligned}
\boldsymbol{\tau}(\tau) \underset{\tau\to 0\mathrm{e}^{\frac{\mathrm{i}\pi\varepsilon_1}{2}}}{=} &\mathrm{const.}\,\tau^{\frac{1}{2}\left(a(a-(-1)^{\varepsilon_2}\mathrm{i})+\frac{1}{4}+8\rho^2\right)}\Big(\mathfrak{p}((-1)^{\varepsilon_2}a,\rho)\chi_1(\vec{\widehat{g}}(\varepsilon_1,\varepsilon_2);\rho)|\tau|^{2\rho}\\
&+\ \mathfrak{p}((-1)^{\varepsilon_2}a,-\rho)\chi_1(\vec{\widehat{g}}(\varepsilon_1,\varepsilon_2);-\rho)|\tau|^{-2\rho}\Big)\left(1+o\left(\tau^\delta\right)\right).
\end{aligned}\tag{A.10}$$

**Conjecture A.2.** *For the conditions stated in Theorem* A.5*,*

$$\boldsymbol{\tau}(\tau) \underset{\tau\to 0\mathrm{e}^{\frac{\mathrm{i}\pi\varepsilon_1}{2}}}{=} \mathrm{const.}\,\tau^{\frac{1}{2}\left(a(a-(-1)^{\varepsilon_2}\mathrm{i})+\frac{1}{4}\right)}\left(\widehat{a}_2(\varepsilon_1,\varepsilon_2)+\widehat{b}_2(\varepsilon_1,\varepsilon_2)\ln|\tau|\right)\left(1+o\left(\tau^\delta\right)\right).\tag{A.11}$$

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